Reduction identities of the minimum rank on loop graphs

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Joint work with Chassidy Bozeman, AnnaVictoria Ellsworth,
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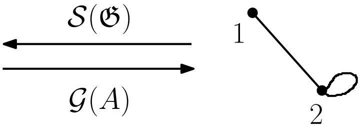
Aaron Rodriguez, and James Strickland.

Minimum Rank of Loop Graphs

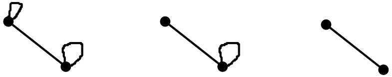
$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \end{array}$$



$$\mathrm{mr}(\mathfrak{G}) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}(\mathfrak{G})\}$$

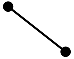
Example: K_2

Loop graph $\mathfrak{G}$



$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$$

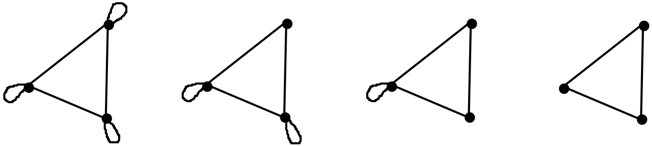
Simple graph G



$$\begin{pmatrix} ? & * \\ * & ? \end{pmatrix}$$

$$\mathrm{mr}(\mathfrak{G}_1) = 1 \quad \mathrm{mr}(\mathfrak{G}_2) = 2 \quad \mathrm{mr}(\mathfrak{G}_3) = 2 \xrightarrow{\min} \mathrm{mr}(G) = 1$$

Example: Complete Graphs

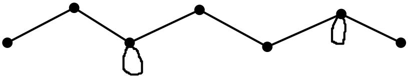


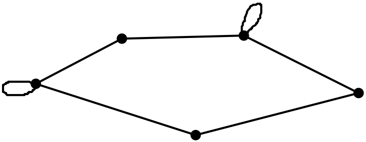
$$\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$$

$$\mathrm{mr}(\mathfrak{G}_1) = 1 \quad \mathrm{mr}(\mathfrak{G}_2) = 2 \quad \mathrm{mr}(\mathfrak{G}_3) = 2 \quad \mathrm{mr}(\mathfrak{G}_4) = 3$$

$$\det \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} = 2abc \neq 0$$

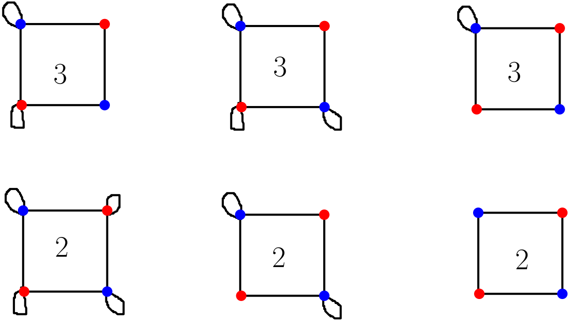
$$n-1=\mathrm{mr}(P_n)\leq \mathrm{mr}(\mathfrak{P}_n)\leq n$$





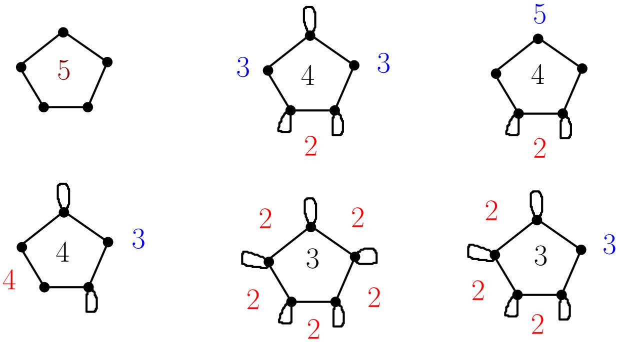
$$n-2=\mathrm{mr}(C_n)\leq \mathrm{mr}(\mathfrak{C}_n)\leq n$$

Even Cycles



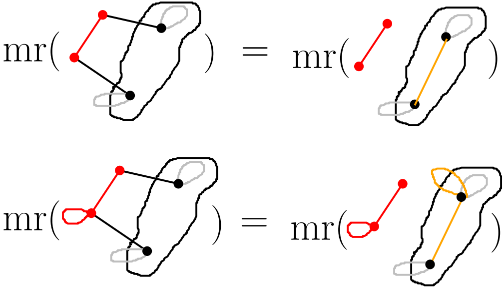
$n-1 \Leftrightarrow$ at least one of blue or red has exactly one loop;
$n-2 \Leftrightarrow$ otherwise.

Odd Cycles



$n \Leftrightarrow$ loopless;
$n-1 \Leftrightarrow$ exactly one even end-loop interval;
$n-2 \Leftrightarrow$ otherwise.

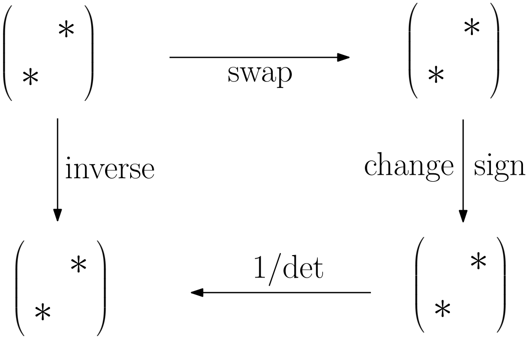
Main Lemma



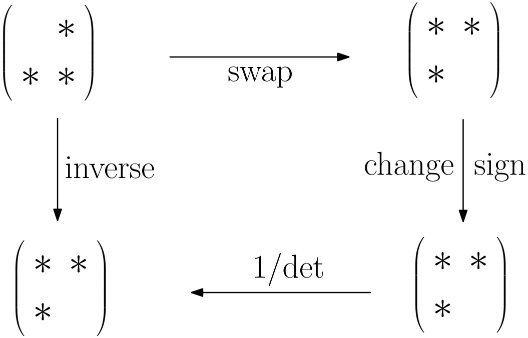
Inverse of 2×2 matrices

$$
\begin{array}{ccc}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \xrightarrow[\text{swap}]{} & \begin{pmatrix} d & b \\ c & a \end{pmatrix} \\
\Big\downarrow \text{inverse} & & \text{change} \Big\downarrow \text{sign} \\
\frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & \xleftarrow{1/\det} & \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\end{array}
$$

Symbolic Inverse

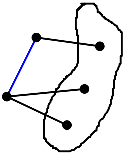


Symbolic Inverse

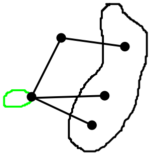


Graph Interpretation

$$\begin{pmatrix} & * & \sim & \sim \\ * & & \sim & \sim \\ \sim & \sim & \sim & \sim \\ \sim & \sim & \sim & \sim \end{pmatrix} \longrightarrow$$



$$\begin{pmatrix} & * & \sim & \sim \\ * & * & \sim & \sim \\ \sim & \sim & \sim & \sim \\ \sim & \sim & \sim & \sim \end{pmatrix} \longrightarrow$$

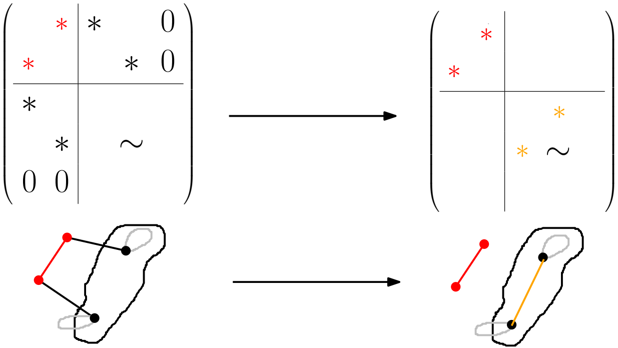


Schur Complement

$$\begin{pmatrix} A & B^\top \\ B & D \end{pmatrix} \xrightarrow{\text{row } 2 - BA^{-1} \text{ row1}} \begin{pmatrix} A & B^\top \\ O & D - BA^{-1}B^\top \end{pmatrix}$$

- If A is invertible, then
  $D - BA^{-1}B^\top$ is called the *Schur complement*.
- Two matrices have the same rank.

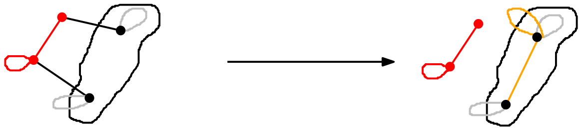
Schur Complement on Graphs



$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

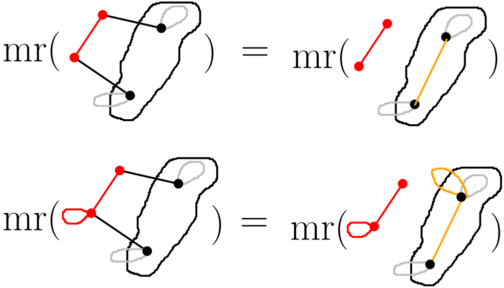
Schur Complement on Graphs

$$\left(\begin{array}{cc|ccc} & \color{red}{*} & * & & 0 \\ \color{red}{*} & \color{red}{*} & & * & 0 \\ \hline * & & & & \\ & * & & \sim & \\ 0 & 0 & & & \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} & \color{red}{*} & & \\ \color{red}{*} & \color{red}{*} & & \\ \hline & & \color{gray}{*}+\color{orange}{*} & \color{orange}{*} \\ & & \color{orange}{*} & \sim \\ & & & \end{array}\right)$$

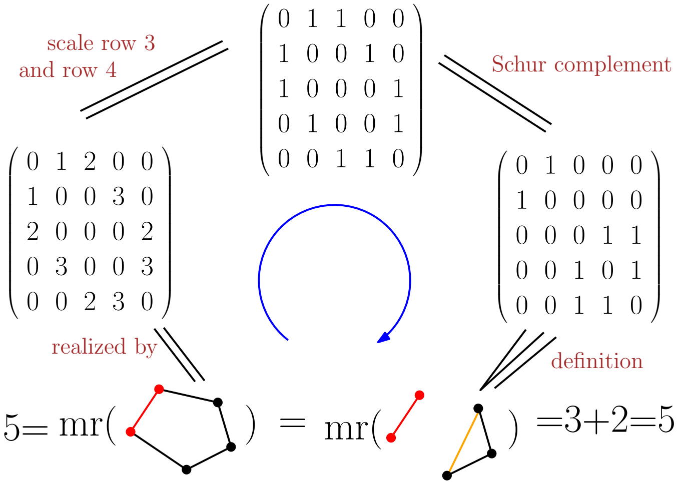


$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Main Lemma



Proof of Main Lemma

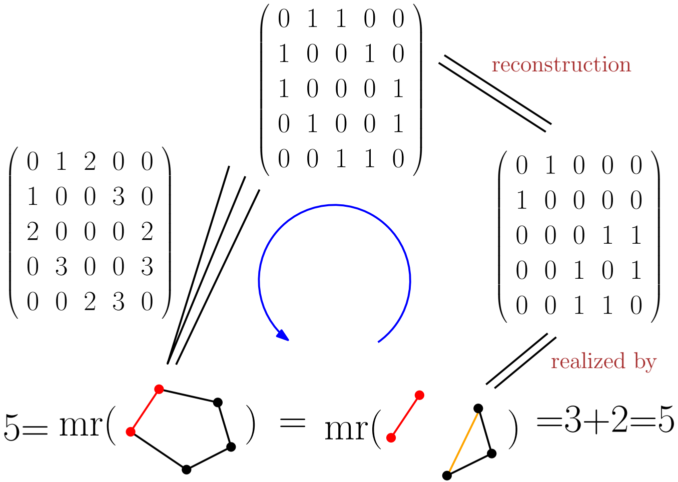


$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

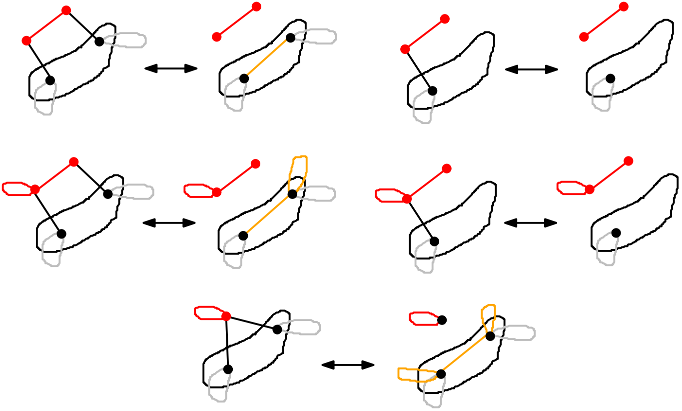
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

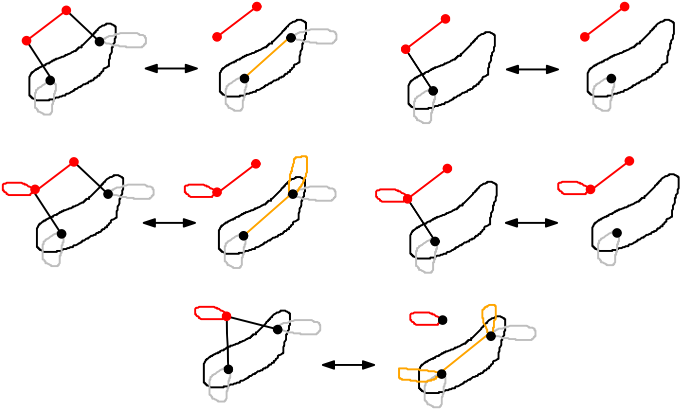
Proof of Main Lemma



Other Results



Other Results



Thank you.