Minimum Rank of Graphs with Loops

Jephian C.-H. Lin, Gabi Maurer, Kathleen Nowak

Iowa State University

April 28, 2014

Outline

Loop Graphs

Definition

- *A loop graph is a pair $\mathfrak{G} = (V, E)$ where V is a set of vertices and E is a set of edges.*
- *Each edge is associated with two vertices (not necessarily distinct) called its endpoints.*
- *A loop is an edge whose endpoints are equal.*

Loop Graphs

Definition

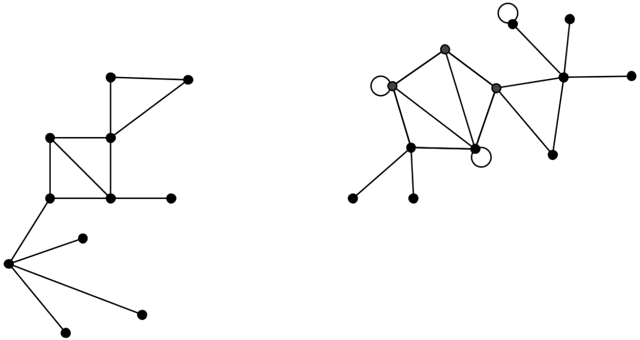
- *A loop graph is a pair $\mathfrak{G} = (V, E)$ where V is a set of vertices and E is a set of edges.*
- *Each edge is associated with two vertices (not necessarily distinct) called its endpoints.*
- *A loop is an edge whose endpoints are equal.*

- We'll denote the edge with endpoints u and v by uv.
- All graphs discussed here are finite.

Loop Graphs

Examples of Loop Graphs

$\mathfrak{G}_1$ $\mathfrak{G}_2$



Loop Graph from a Real Symmetric Matrix

Definition

Let $A \in \mathbb{R}^{n \times n}$ *with* $A^T = A$.

The loop graph $\mathcal{G}_\ell(A) = (V, E)$ *is the graph with*

- $V = \{1, 2, ..., n\}$
- $E = \{ij \mid a_{ij} \neq 0\}$

Loop Graph from a Real Symmetric Matrix

Definition

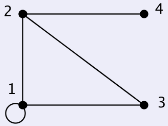
Let $A \in \mathbb{R}^{n \times n}$ *with* $A^T = A$.

The *loop graph* $\mathcal{G}_\ell(A) = (V, E)$ *is the graph with*

- $V = \{1, 2, \ldots, n\}$
- $E = \{ij \mid a_{ij} \neq 0\}$

Example

$$A = \begin{bmatrix} -1 & 2 & 5 & 0 \\ 2 & 0 & 1 & -3 \\ 5 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \Rightarrow$$



$\mathcal{G}_\ell(A)$

Minimum Rank of a Loop Graph

Definition

Given a loop graph $\mathfrak{G}$, we define

$$\mathcal{S}_\ell(\mathfrak{G}) := \{A \in \mathbb{R}^{n\times n} \mid A^T = A,\ \mathcal{G}_\ell(A) = \mathfrak{G}\}$$

Minimum Rank of a Loop Graph

Definition

Given a loop graph $\mathfrak{G}$, we define

$$\mathcal{S}_\ell(\mathfrak{G}) := \{A \in \mathbb{R}^{n\times n} \mid A^T = A,\ \mathcal{G}_\ell(A) = \mathfrak{G}\}$$

Definition

The minimum rank of a loop graph $\mathfrak{G}$ is defined to be

$$\mathrm{mr}(\mathfrak{G}) := \min\{\mathit{rank}\ A \mid A \in \mathcal{S}_\ell(\mathfrak{G})\}$$

Questions Asked

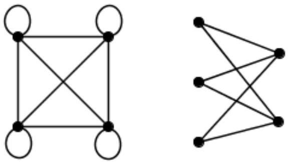
Questions Posed

1. Given a loop graph $\mathfrak{G}$,
   - When is $\text{mr}(\mathfrak{G}) \leq 2$?
   - When is $\text{mr}(\mathfrak{G}) = |G|$?
2. Can we determine the minimum rank for certain families of loop graphs ($\mathfrak{K}_n, \mathfrak{P}_n, \mathfrak{C}_n$, etc)?

Definition and Notation

Definition

- $\mathfrak{K}_n$ *is the complete graph on n vertices with all loops*
- $\mathfrak{K}_{s,t}$ *is the complete bipartite graph without loops*
- $\dot{\cup}$ *denotes the disjoint union of two graphs*
- $\vee$ *denotes the join of two graphs*

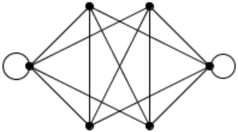


$\mathfrak{K}_4$ $\mathfrak{K}_{3,2}$

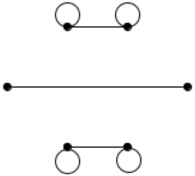
Definition and Notation

Definition

Let $\mathfrak{G}$ be a loop graph. The complement of $\mathfrak{G}$ is the graph $\overline{\mathfrak{G}} = (V, \overline{E})$ where $E \cap \overline{E} = \emptyset$ and for $v_i, v_j \in V$ if an edge or a loop $v_i v_j \notin E$, then $v_i v_j \in \overline{E}$.



$(\mathfrak{K}_1 \dot\cup \mathfrak{K}_1) \vee (\mathfrak{K}_{2,2})$



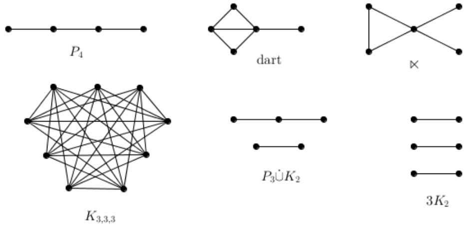
$\mathfrak{K}_{1,1} \dot\cup (\mathfrak{K}_2 \dot\cup \mathfrak{K}_2)$

mr ≤ 2 for simple graphs

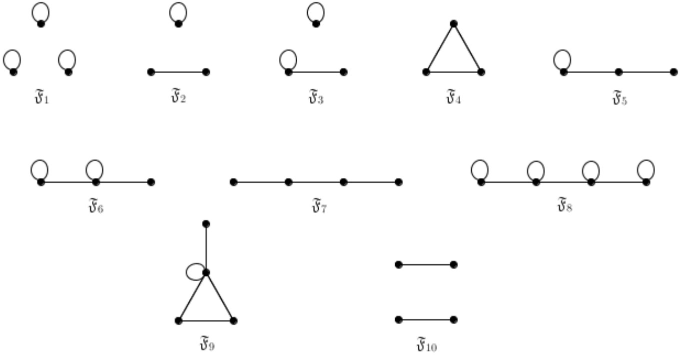
Theorem (Barrett, van der Holst, Loewy, 2004)

Let G be a simple graph. The following are equivalent:

1. $\text{mr}(G) \leq 2$.
2. *G is $\{P_4, \text{dart}, \ltimes, K_{3,3,3}, P_3 \dot{\cup} K_2, 3K_2\}$-free.*
3. $\overline{G} = (K_{s_1} \dot{\cup} K_{s_2} \dot{\cup} K_{p_1,q_1} \dot{\cup} \cdots \dot{\cup} K_{p_k,q_k}) \vee K_r$ *for some nonnegative s_1, s_2, k, p_i, q_i, r with $p_i + q_i \geq 1$ for $i = 1, \ldots, k$.*



mr $\leq$ 2 for loop graphs



Set $\mathcal{F}$ of forbidden induced subgraphs

$\mathrm{mr} \leq 2$ for loop graphs

Theorem

Let $\mathfrak{G}$ be a loop graph. The following are equivalent:

1. $\mathrm{mr}(\mathfrak{G}) \leq 2$.
2. $\mathfrak{G}$ *is* $\mathcal{F}$*-free for the set* $\mathcal{F}$ *of loop graphs*
3. $\overline{\mathfrak{G}} = (\mathfrak{K}_{s_1} \dot{\cup} \mathfrak{K}_{s_2} \dot{\cup} \mathfrak{K}_{p_1,q_1} \dot{\cup} \cdots \dot{\cup} \mathfrak{K}_{p_k,q_k}) \vee \mathfrak{K}_r$ *for some nonnegative* s_1, s_2, k, p_i, q_i, r *with* $p_i + q_i \geq 1$ *for* $i = 1, \ldots, k$.
4. $\mathfrak{G} = (\mathfrak{K}_{s_1,s_2} \vee (\mathfrak{K}_{p_1} \dot{\cup} \mathfrak{K}_{q_1}) \vee \cdots \vee (\mathfrak{K}_{p_k} \dot{\cup} \mathfrak{K}_{q_k})) \dot{\cup} \overline{\mathfrak{K}_r}$ *for some nonnegative* s_1, s_2, k, p_i, q_i, r *with* $p_i + q_i \geq 1$ *for* $i = 1, \ldots, k$.

Sketch of the Proof I

3 $\Leftrightarrow$ 4 $\Rightarrow$ 1 $\Rightarrow$ 2: $\checkmark$

Sketch of the Proof I

(3) $\Leftrightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2): $\checkmark$

(2) $\Rightarrow$ (3):

Consider a $\mathcal{F}$-free loop graph $\mathfrak{G}$

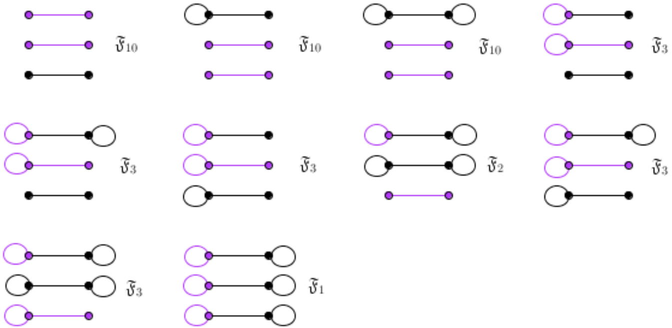
Sketch of the Proof I

③ $\Leftrightarrow$ ④ $\Rightarrow$ ① $\Rightarrow$ ②: ✓

② $\Rightarrow$ ③:

Consider a $\mathcal{F}$-free loop graph $\mathfrak{G}$

$\mathfrak{G}$ is $\{P_4$, dart, $\ltimes$, $K_{3,3,3}, P_3\dot{\cup}K_2, 3K_2\}$-free



Sketch of the Proof II

The complement of the underlying simple graph of $\mathfrak{G}$ is

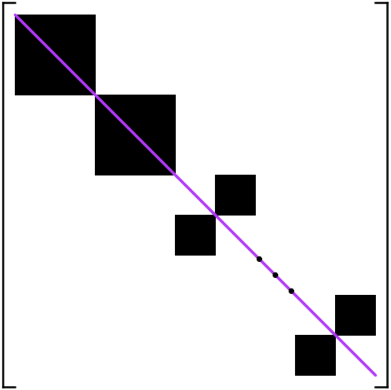
$$\overline{G} = \left(K_{s_1} \dot{\cup} K_{s_2} \dot{\cup} K_{p_1,q_1} \dot{\cup} \cdots \dot{\cup} K_{p_k,q_k}\right) \vee K_r$$

Sketch of the Proof II

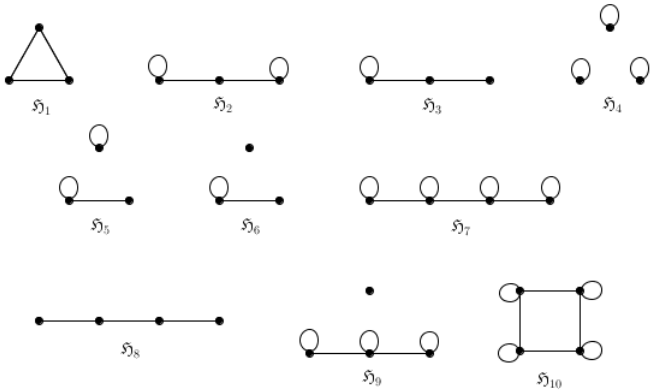
The complement of the underlying simple graph of $\mathfrak{G}$ is

$$\overline{G} = \left(K_{s_1} \dot\cup K_{s_2} \dot\cup K_{p_1,q_1} \dot\cup \cdots \dot\cup K_{p_k,q_k}\right) \vee K_r$$

We do not have to consider the join with K_r.



Sketch of the Proof III

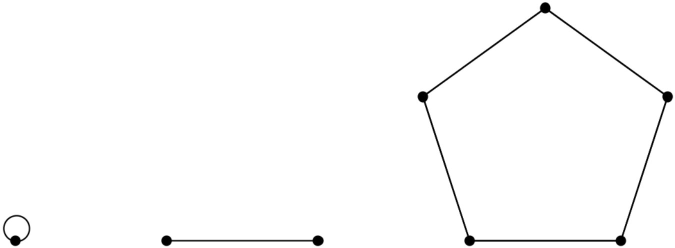


Complements of the forbidden subgraphs

Generalized Cycles

Definition

A generalized cycle of a loop graph $\mathfrak{G}$ is a subgraph of $\mathfrak{G}$ whose connected components are either loops, single edges, or cycles.



Generalized Cycles

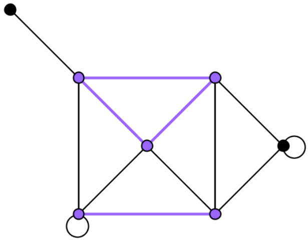
Definition

The order of a generalized cycle is the number of vertices in the generalized cycle. A generalized cycle of order $|\mathfrak{G}|$ *is said to be spanning.*

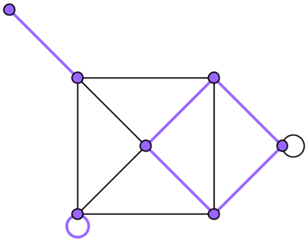
Generalized Cycles

Definition

The order *of a generalized cycle is the number of vertices in the generalized cycle. A generalized cycle of order* $|\mathfrak{G}|$ *is said to be* spanning.



generalized cycle of order 5



spanning generalized cycle

Generalized Cycles

Definition

Given a generalized cycle $\mathcal{C}$, define

- $nc(\mathcal{C})$:= *the number of distinct cycles in $\mathcal{C}$, and*
- $ne(\mathcal{C})$:= *the number of even components of $\mathcal{C}$*

For a loop graph $\mathfrak{G}$, define

- $cyc_k(\mathfrak{G})$:= *the set of all generalized cycles of order k.*

Generalized Cycles

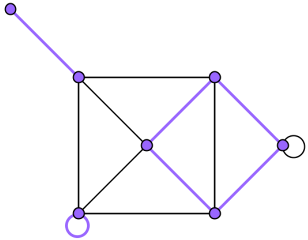
Definition

Given a generalized cycle $\mathcal{C}$, define

- $nc(\mathcal{C})$:= *the number of distinct cycles in* $\mathcal{C}$, *and*
- $ne(\mathcal{C})$:= *the number of even components of* $\mathcal{C}$

For a loop graph $\mathfrak{G}$, *define*

- $cyc_k(\mathfrak{G})$:= *the set of all generalized cycles of order* k.



$$\mathbf{nc}(\mathcal{C}) = \mathbf{1}$$

$$\mathbf{ne}(\mathcal{C}) = \mathbf{2}$$

Permutation Association

With $\mathcal{C}$, we can associate a permutation of the vertices as follows:

- For each cycle, fix an orientation and associate the directed graph cycle $(v_1, v_2, \ldots, v_k)$ with the cyclic permutation $(v_1, v_2, \ldots v_k)$.
- Associate each edge $v_1 v_2$ with the 2-cycle $(v_1 v_2)$.
- Associate each loop $v_1 v_1$ with the permutation (v_1).

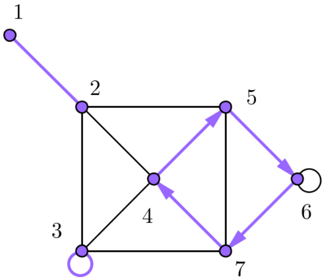
Permutation Association

With $\mathcal{C}$, we can associate a permutation of the vertices as follows:

- For each cycle, fix an orientation and associate the directed graph cycle $(v_1, v_2, \ldots, v_k)$ with the cyclic permutation $(v_1, v_2, \ldots v_k)$.
- Associate each edge $v_1 v_2$ with the 2-cycle $(v_1 v_2)$.
- Associate each loop $v_1 v_1$ with the permutation (v_1).

The permutation $\pi_{\mathcal{C}}$ is then defined to be the product of these associated permutation cycles.

Permutation Association



$$\pi_{\mathcal{C}} = (\mathbf{12})(\mathbf{3})(\mathbf{4567})$$

Generalized Cycles and Determinant

Theorem (Harary, 1962)

For $A = [a_{ij}] \in \mathcal{S}(\mathfrak{G})$, let $S_k(A)$ denote the sum of all $k \times k$ principal minors. Then

$$S_k(A) = \sum_{\mathcal{C} \in cyc_k(\mathcal{G}_\ell(A))} (-1)^{ne(\mathcal{C})} 2^{nc(\mathcal{C})} a_{i_1 \pi_{\mathcal{C}}(i_1)} \cdots a_{i_k \pi_{\mathcal{C}}(i_k)}$$

Generalized Cycles and Determinant

Theorem (Harary, 1962)

For $A = [a_{ij}] \in \mathcal{S}(\mathfrak{G})$, let $S_k(A)$ denote the sum of all $k \times k$ principal minors. Then

$$S_k(A) = \sum_{\mathcal{C} \in cyc_k(\mathcal{G}_\ell(A))} (-1)^{ne(\mathcal{C})} 2^{nc(\mathcal{C})} a_{i_1 \pi_{\mathcal{C}}(i_1)} \cdots a_{i_k \pi_{\mathcal{C}}(i_k)}$$

In particular,

$$\det(A) = S_n(A) = \sum_{\mathcal{C} \in \mathrm{cyc}_n(\mathcal{G}_\ell(A))} (-1)^{\mathrm{ne}(\mathcal{C})} 2^{\mathrm{nc}(\mathcal{C})} a_{i_1 \pi_{\mathcal{C}}(i_1)} \cdots a_{i_n \pi_{\mathcal{C}}(i_n)}$$

Auxiliary Result

Lemma

If $p(x_1, x_2, \ldots, x_q)$ is a nonzero homogeneous polynomial over $\mathbb{R}$, then there exist nonzero real numbers $c_1, c_2, \ldots, c_q$ such that $p(c_1, c_2, \ldots, c_q) \neq 0$.

High Minimum Rank Characterization

Theorem

For every loop graph $\mathfrak{G}$, $\mathrm{mr}(\mathfrak{G}) = |\mathfrak{G}|$ *if and only if* $\mathfrak{G}$ *has a unique spanning generalized cycle.*

Proof Sketch:

($\Leftarrow$) Clear.

High Minimum Rank Characterization

($\Rightarrow$) Now assume $mr(\mathfrak{G}) = |\mathfrak{G}|$.

If $\mathfrak{G}$ is a loopless, then $\text{mr}(\mathfrak{G}) = \text{mr}_0(\mathfrak{G})$ and the result has been proven.

Otherwise, if $\mathfrak{G}$ contains a loop uu, we proceed by contradiction.

High Minimum Rank Characterization

Consider a minimal counterexample $\mathfrak{H}^* = (V^*, E^*)$.
For a loop graph $\mathfrak{H} = (V, E)$,

1. If $|V| < |V^*|$ and $\mathrm{mr}(\mathfrak{H}) = |V| \Rightarrow \mathfrak{H}$ has a unique spanning generalized cycle.
2. If $|V| = |V^*|$, $|E| < |E^*|$, and $\mathrm{mr}(\mathfrak{H}) = |V| \Rightarrow \mathfrak{H}$ has a unique spanning generalized cycle.

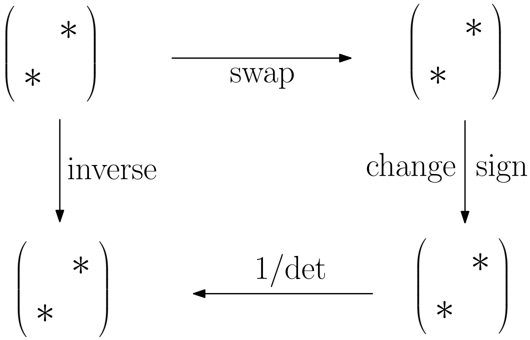
Cases:

1. uu participates in every spanning generalized cycle.
2. uu participates in no spanning generalized cycle.
3. uu participates in the cycle $\mathcal{C}_1$ but not in $\mathcal{C}_2$.

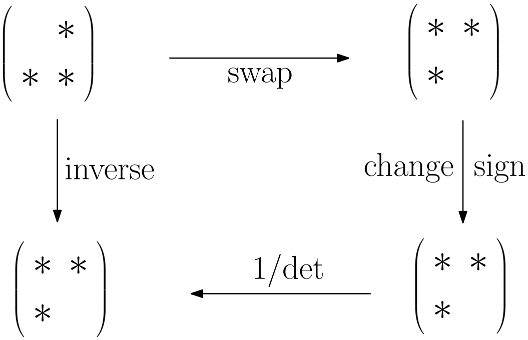
Inverse of 2×2 matrices

$$
\begin{array}{ccc}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} & \xrightarrow[\text{swap}]{} & \begin{pmatrix} d & b \\ c & a \end{pmatrix} \\
\Big\downarrow \text{inverse} & & \text{change} \Big\downarrow \text{sign} \\
\frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} & \xleftarrow{1/\det} & \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\end{array}
$$

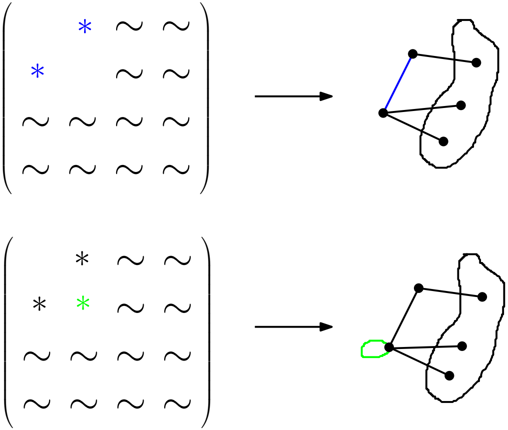
Symbolic Inverse



Symbolic Inverse



Graph Interpretation



Schur Complement

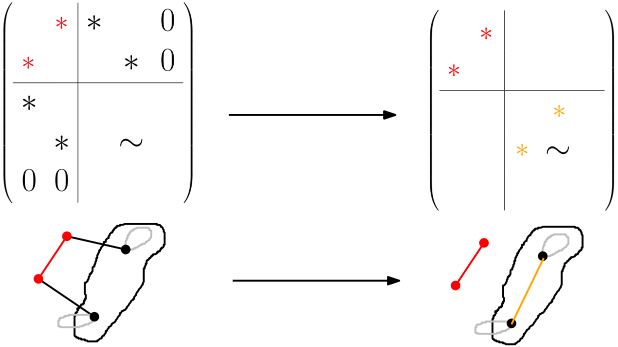
$$\begin{pmatrix} A & B^\top \\ B & D \end{pmatrix} \xrightarrow{\text{row 2} - BA^{-1}\text{ row1}} \begin{pmatrix} A & B^\top \\ O & D - BA^{-1}B^\top \end{pmatrix}$$

- If A is invertible, then

  $D - BA^{-1}B^\top$ is called the *Schur complement*.

- Two matrices have the same rank.

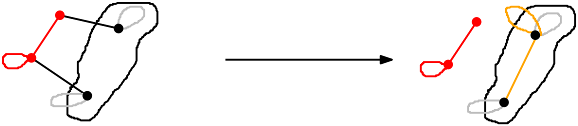
Schur Complement on Graphs



$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

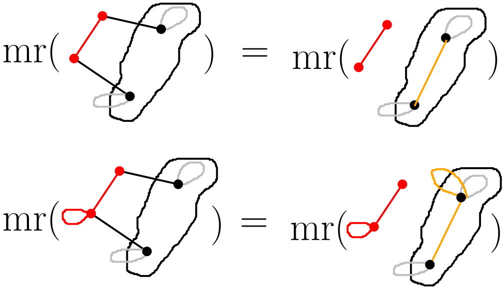
Schur Complement on Graphs

$$\left(\begin{array}{cc|ccc} & {\color{red}*} & * & & 0 \\ {\color{red}*} & {\color{red}*} & & * & 0 \\ \hline * & & & & \\ & * & & \sim & \\ 0 & 0 & & & \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} & {\color{red}*} & & \\ {\color{red}*} & {\color{red}*} & & \\ \hline & & {\color{gray}*}+{\color{orange}*} & {\color{orange}*} \\ & & {\color{orange}*} & \sim \\ & & & \end{array}\right)$$



$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Main Lemma



Proof of Main Lemma

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

scale row 3 and row 4

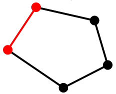
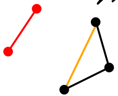
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix}$$

Schur complement

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

realized by

definition

$5 = \mathrm{mr}($  $) = \mathrm{mr}($  $) = 3+2=5$

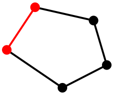
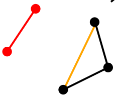
Proof of Main Lemma

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

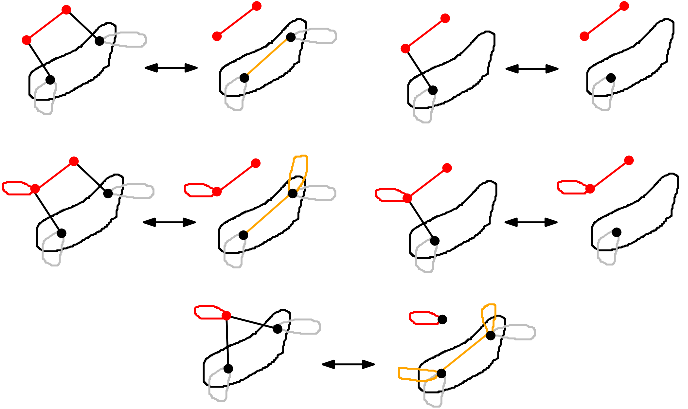
reconstruction

$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

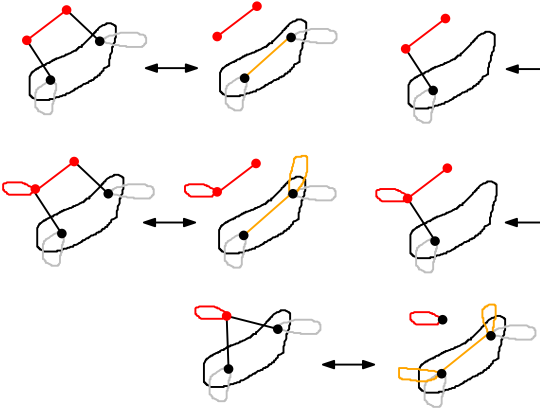
realized by

$5 = \mathrm{mr}($  $) = \mathrm{mr}($  $) = 3 + 2 = 5$

Other Results



Other Results



Thank you.

References I

W.W. Barrett, H. van der Holst, and R. Loewy. Graphs whose minimal rank is two. *Elec. J. Lin. Alg.*, 11: 258-280, 2004.

C. Grood, A. Harmse, L. Hogben, T.J. Hunter, B. Jacob, A. Klimas, S. McCathern. Minimum rank with zero diagonal. Under review.

F. Harary. The determinant of the adjacency matrix of a graph. *SIAM Review*, 4: 202-210, 1962.