Odd cycle zero forcing parameters and the minimum rank problem

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Minimum rank problem (simple and loop)

$$\begin{bmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{bmatrix}$$

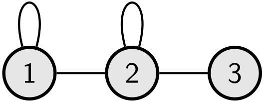
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\mathrm{mr}(P_3) = 2$

$M(P_3) = 1$

smallest possible rank

largest possible nullity



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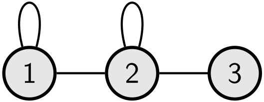
$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\mathrm{mr}(\mathfrak{P}_3) = 3$

$M(\mathfrak{P}_3) = 0$

Zero forcing number (simple and loop)



$M(P_3) = 1$
$Z(P_3) = 1$

$M(\mathfrak{P}_3) = 0$
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minimum number of blue vertices
to force all vertices blue

simple If y is the only white neighbor of x and x is blue, then $x \to y$.

loop If y is the only white neighbor of x ~~and x is blue~~, then $x \to y$. (x, y are possibly the same.)

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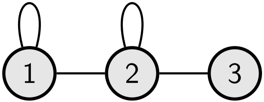

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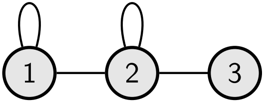

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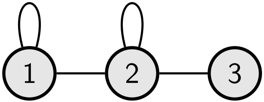

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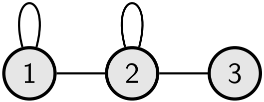

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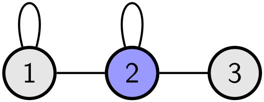

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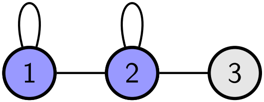

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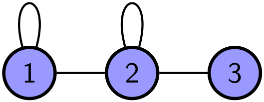

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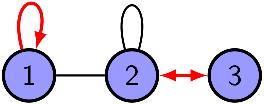
Max Nullity vs Zero Forcing

- $M(G) \leq Z(G)$ for all simple graph [AIM 2008]; $M(\mathfrak{G}) \leq Z(\mathfrak{G})$ for all loop graph [Hogben 2010].
- By definition, $M(G) = \max_{\mathfrak{G}} M(\mathfrak{G})$ over all loop configurations.
- Define $\widehat{Z}(G) = \max_{\mathfrak{G}} Z(\mathfrak{G})$ over all loop configurations. Then

$$M(G) \leq \widehat{Z}(G) \leq Z(G).$$

Proof of $M \leq Z$



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix} \end{array}$$

$3 \to 2$
$1 \to 1$
$2 \to 3$

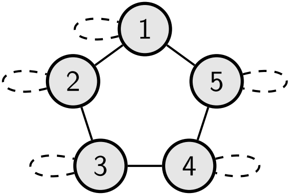
$$\begin{array}{c} \\ 2 \\ 1 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \end{array}$$

Max Nullity vs Zero Forcing (Conti.)

- $M(G) = Z(G)$ whenever $|V(G)| \leq 7$ or G is a tree, a cycle; not always true for outerplanar graphs.
- $M(\mathfrak{G}) = Z(\mathfrak{G})$ whenever $|V(\mathfrak{G})| \leq 2$ or $\mathfrak{G}$ is a loop configuration of a tree; not always true for cycles or outerplanar graphs.
- For the simple graph $K_{3,3,3}$, $M(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = 7$.
- For the loop graph $\mathfrak{C}_3^0$, $M(\mathfrak{C}_3^0) = 0$ and $Z(\mathfrak{C}_3^0) = 1$.
- The fact $M^F(\mathfrak{C}_{2k+1}^0) = 0$ is true whenever the considered matrix is symmetric and char $\neq 2$.

An example with $M(\mathfrak{G}) \neq Z(\mathfrak{G})$

- $M(\mathfrak{C}_n) = Z(\mathfrak{C}_n)$ if $\mathfrak{C}_n$ is not a loopless odd cycle; $M(\mathfrak{C}^0_{2k+1}) = 0$ but $Z(\mathfrak{C}^0_{2k+1}) = 1$.
- $M^{\mathbb{R}}(\mathfrak{C}^0_{2k+1}) = 0$ but $M^{\mathbb{F}_2}(\mathfrak{C}^0_{2k+1}) = 1$.



$$\det \begin{bmatrix} 0 & e_1 & & & e_{2k+1} \\ e_1 & 0 & e_2 & & \\ & e_2 & \ddots & \ddots & \\ & & \ddots & & e_{2k} \\ e_{2k+1} & & & e_{2k} & 0 \end{bmatrix} = 2\prod_{i=1}^{2k+1} e_i$$

Try to generalize the "triangle"

$$\text{rank}\begin{bmatrix} a_{1,1} & ? & ? & ? & ? \\ 0 & a_{2,2} & ? & ? & ? \\ 0 & 0 & a_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 3$$

Try to generalize the "triangle"

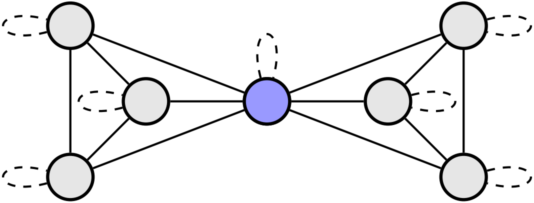
$$\operatorname{rank}\begin{bmatrix} A_{1,1} & ? & ? & ? & ? \\ O & A_{2,2} & ? & ? & ? \\ O & O & A_{3,3} & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq \sum_{i=1}^{3} \operatorname{rank}(A_{i,i})$$

Try to generalize the "triangle"

$$\text{rank}\begin{bmatrix} A(\mathfrak{C}_5^0) & ? & ? & ? & ? \\ O & A(\mathfrak{C}_7^0) & ? & ? & ? \\ O & O & A(\mathfrak{C}_3^0) & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \geq 5 + 7 + 3 = 15$$

Odd cycle zero forcing number

- The color-change rule for loop graphs is:
  - if y is the only white neighbor of x ~~and x is blue~~, then $x \to y$. (x, y are possibly the same.)
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.



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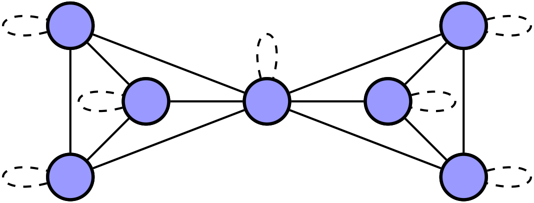

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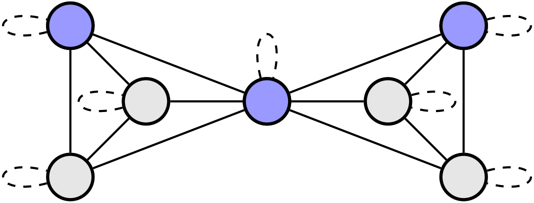

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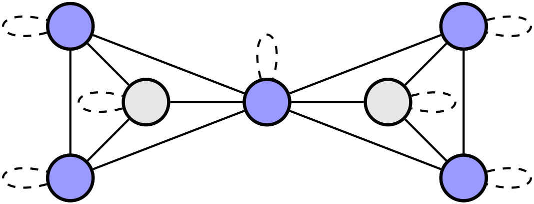

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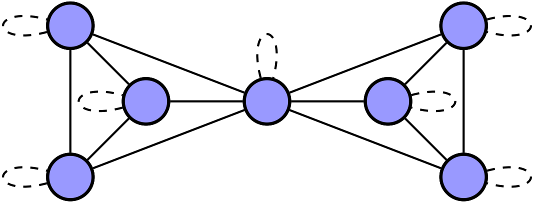

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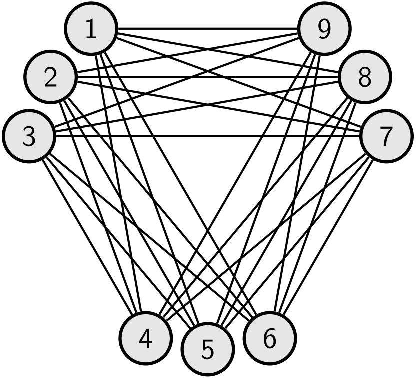
  - if W is the set of white vertices, and $\mathfrak{G}[W]$ has a connected component $\mathfrak{C}$ such that $\mathfrak{C} \cong \mathfrak{C}^0_{2k+1}$, then all vertices in $V(\mathfrak{C})$ turn blue.
- $Z_{oc}(\mathfrak{G})$ is the minimum number of blue vertices required to force all graph blue.
- $M^F(\mathfrak{G}) \leq Z_{oc}(\mathfrak{G})$ whenever char $F \neq 2$ and matrices are symmetric.
- The enhanced odd cycle zero forcing number is defined as $\widehat{Z}_{oc}(G) = \max_{\mathfrak{G}} Z_{oc}(\mathfrak{G})$, where $\mathfrak{G}$ runs over all loop configurations of G.

Max Nullity vs Zero Forcing Revisit

- $M(G) \leq Z(G)$ for all simple graph [AIM 2008];
  $M(\mathfrak{G}) \leq Z(\mathfrak{G})$ for all loop graph [Hogben 2010].
- For simple graphs with $|V(G)| \leq 7$, $M(G) = Z(G)$.
- For the simple graph $K_{3,3,3}$, $M(K_{3,3,3}) = 6$ and $\widehat{Z}(K_{3,3,3}) = 7$.
- For the loop graph $\mathfrak{C}_3^0$, $M(\mathfrak{C}_3^0) = 0$ and $Z(\mathfrak{C}_3^0) = 1$.
- $M(K_{3,3,3}) = 6 = \widehat{Z}_{oc}(K_{3,3,3})$.
- $M(\mathfrak{C}_{2k+1}^0) = 0 = Z_{oc}(\mathfrak{C}_{2k+1}^0)$.

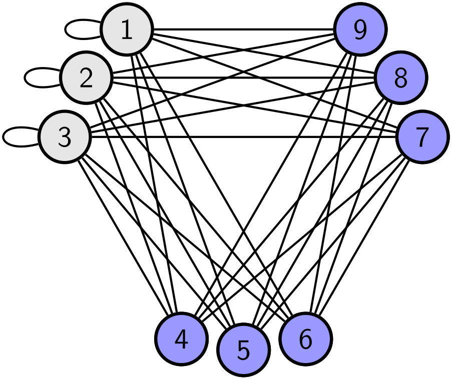
Example: $K_{3,3,3}$



$$\widehat{Z}_{oc}(K_{3,3,3}) = 6 \text{ and } \widehat{Z}(K_{3,3,3}) = Z(K_{3,3,3}) = 7.$$

Example: $K_{3,3,3}$

1,2,3 have loops
others are unknown



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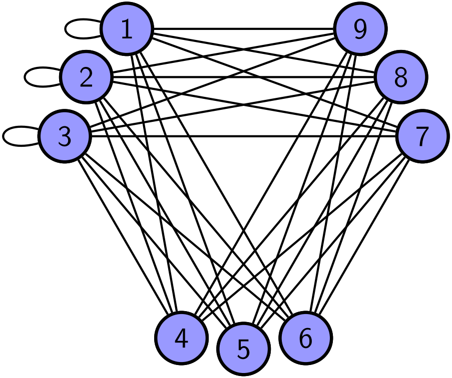

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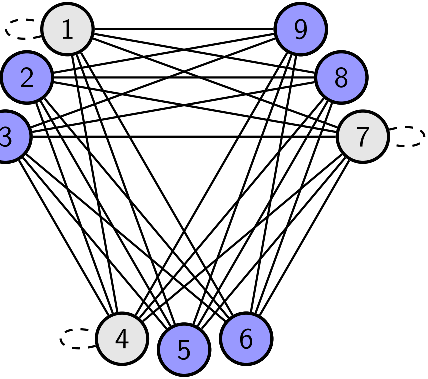
Example: $K_{3,3,3}$

1,4,7 have no loops
others are unknown



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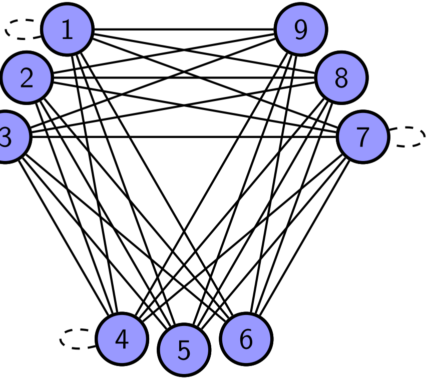

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Field matters

- Let A be the adjacency matrix.
- $\text{null}(A) = 6 = M(K_{3,3,3}) = \widehat{Z}_{oc}(K_{3,3,3})$.
- $\text{null}^{\mathbb{F}_2}(A) = 7 = M^{\mathbb{F}_2}(K_{3,3,3}) = Z(K_{3,3,3})$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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