Variants of Zero Forcing (ZF)

- ZF of patterns: Z, Z^-, Z_ℓ
  - symmetry: Z_{oc}
  - ? = 0 or $*$: $\hat{Z}$
  - minor-monotone: $\lfloor Z \rfloor$
- ZF of sign patterns: $Z_\pm$
- ZF for inertias: Z_q.
  $Z_+ = Z_0 \leq Z_1 \leq \cdots \leq Z_n = Z$
- ZF for Strong Arnold Property: Z_{SAP}

ZF on a graph G.

- vertices: blue or white.
- pick some vtx blue
- color-change rule (CCR)
- zero forcing number
  $Z(G)$: min blue vtx
  that can make whole graph blue

ZF of patterns:

pattern = matrix over $\{0, *, ?\}$

$*$: nonzero; ?: any element.

$$\begin{array}{c|c} \text{eq'n} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \end{pmatrix} \end{array}$$

max nullity = 0

ZF: growing knowledge.

variables: blue or white

blue: zero; white: uncertain.

eq'n 4 $\Rightarrow x_3 = 0$

eq'n 1 $\Rightarrow x_2 = 0$

eq'n 2 $\Rightarrow x_1 = 0$

eq'n 3 $\Rightarrow x_4 = 0$

$$\begin{array}{c|c} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \end{array}$$

$\boxed{\text{Assume: } x_4 = 0}$

eq'n 4 $\Rightarrow x_3 = 0$

eq'n 1 $\Rightarrow x_2 = 0$

eq'n 2 $\Rightarrow x_1 = 0$

First 3 columns are indep.

$\Rightarrow$ max nullity $\leq \underset{\sim}{1}$

$\sim$ initial blue variables.

$$\begin{array}{c|c} & \begin{array}{cccc} 3 & 2 & 1 & 4 \end{array} \\ \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} * & & & * \\ * & * & & \\ * & * & * & \\ * & * & * & * \end{pmatrix} \end{array}$$

Patterns / Graphs.

	sym	not sym
free diagonal	$\begin{pmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{pmatrix}$ simple	$\begin{pmatrix} ? & * & 0 \\ 0 & ? & * \\ 0 & 0 & ? \end{pmatrix}$ simple digraph
not free	$\begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & 0 \end{pmatrix}$ loop	$\begin{pmatrix} 0 & * & 0 \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}$ loop digraph

Thm: $M(G) \le Z(G)$ [AIM'08, H'10]

$\boxed{\text{BG'07 Giovannetti}}$

ZF on graphs: Z.

CCR:

simple: For vtx i, if j is the only white neighbor and i is blue $\Rightarrow i \to j$

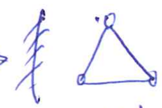
loop: For vtx i, if j is the only white neighbor $\Rightarrow i \to j$

digraph: nbr $\to$ out-nbr

Symmetry:

G: simple or loop graph.

$M(G)$: ~~focus on symmetric mtx.~~ max nullity over sym mtx of G.

$\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \longleftrightarrow$ 

$P \downarrow$ $M(P)=1$

$G \downarrow$ $\det\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} = 2abc \ne 0$ $M(G)=0$

Odd cycle ZF: Z_{oc} (loop graph).

CCR-Z_{oc}:

- CCR-Z.
- B = blue vtx
  if a component of $G \setminus B$ is a loopless odd cycle C,
  $\Rightarrow$ ~~VCC~~ $V(C)$ turn blue.

Thm: [L'16]
$M(G) \le Z_{oc}(G) \le Z(G)$
G: loop graphs.

? = * or 0

simple graph | loop configurations

By def, $\underset{\text{simple}}{M(G)} = \max\limits_{\mathcal{G}} M(\mathcal{G})$ ← all loop ~~fo~~ config's

Thm: [BBFHHSvdDvdH ~~'1~~'13, L'16]
G: simple graph

$M(G) \le \max\limits_{\mathcal{G}} Z_{oc}(\mathcal{G}) \le \max\limits_{\mathcal{G}} Z(\mathcal{G}) \le Z(G)$.

loop config's

$\max\limits_{\mathcal{G}} Z_{oc}(\mathcal{G}) = \hat{Z}_{oc}$, $\max\limits_{\mathcal{G}} Z(\mathcal{G}) = \hat{Z}$

Minor-monotone:

$\beta(H) \le \beta(G)$ if H is a minor of G.

Colin de Verdièr type parameter

$\xi(G) \le M(G) \le Z(G)$.
↑ minor-monotone

Minor-monoton floor of β

$\lfloor\beta\rfloor(H) = \min\{\beta(G) : H \text{ is a minor of } G\}$.

$\xi(G) \le \lfloor M\rfloor(G) \le \lfloor Z\rfloor(G)$

Thm: [BBFHHSvdDvdH '13]
CCR-$\lfloor Z\rfloor$:

- CCR-Z
- if i and all nbrs are blue and i hasn't make a force,
  ~~then~~ $\Rightarrow i \to j$ (pick a j)

ZF for sign patterns [GB'14]

sign pattern : matrix over $\{0, ?, +, -\}$.
? : any, + : >0, − : <0

Sign ZF : $Z_{\pm}$.

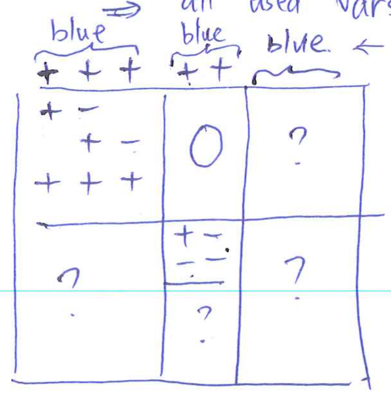
vertices : blue , white < + ≥0 ; − ≤0
blue : zero ; white : uncertain

CCR-$Z_{\pm}$:

- CCR-Z.
- if all white vtxs has no mark (+,−)
  pick one and ~~mat~~ mark it "+"
- $(+|+) + (+|\ ) + (-|-) = 0$
  coef, var ; $(+|\ )$: mark it "−"
- $(+|+) + (+|+) + (-|-) = 0.$
  ≥0 ≥0 ≥0
  ⇒ all used vars turn blue.

e.g.

blue, blue, blue. ← assumption.



mark "+"

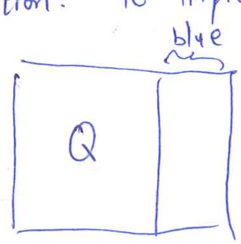
$(+|+) + (-|\ ) = 0$

$(+|+) + (+|+) + (+|+) = 0$
≥0 ≥0 ≥0

$(-|+) + (-|+) = 0$
≤0 ≤0

Thm : max nullity ≤ # initial blue vars.

Question: To improve $Z_{\pm}$.



⇒ Q^T is <u>L-matrix</u>

L-matrix : a sign pattern s.th
every realization has row full-rank.

[ ]

ZF for inertia: Z_q [BGH '15]

Thm: $M_q(G) \le Z_q(G)$

G : simple graph.

Consider ~~Ma~~ mtx with $\le q$ neg eigenvalues.

"Game" on simple graph G.

players: X and Y

X : color all vtx blue.

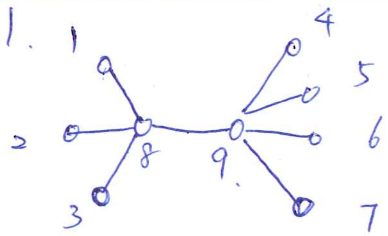
Y : make X use max token

CCR-Z_q:

- X can use $\boxed{1 \text{ token}}$ to color any vtx blue.
- X can apply CCR-Z on G of free
- { B: blue vtx
  G\B has compn'ts $W_1, \dots, W_k$.
  X can pick $\ge q+1$ compn'ts to Y.
  Y return ≥ 1 compn'ts $W_{a_1}, W_{a_2}, \dots, W_{a_s}$.
  X can apply CCR-Z on $G[B \cup W_{a_1} \cup W_{a_2} \cup \dots \cup W_{a_s}]$ for free

Z_q: min tokens required.

e.g. $q = 1$.



9 tokens is possible.
$|V(G)|$

5 tokens is possible.
$Z(G)$

4 tokens is possible.

blue.

2 tokens $\longrightarrow$ 3, 7

CCR-Z $\longrightarrow$ 8, 9

$X \xrightarrow{1,4} Y$: $\xrightarrow{1}$ 1 ✓; $\xrightarrow{4}$ 4 ✗; $\xrightarrow{1,4}$ 1, 4 ✗

$X \xrightarrow{2,4} Y \xrightarrow{2} 2$

2 tokens $\longrightarrow$ 4, 5

CCR-Z $\longrightarrow$ 6

Remark: ① Y might not want to return all compn'ts
② X might not want to spend all token initially

Thm: $Z_+ = Z_0 \le Z_1 \le Z_2 \le \dots \le Z_n = Z$

Question: Combine Z_1, Z_+ $\rightarrow$ Z_μ?

$\mu \leftarrow$ 1 neg eigen sign pattern