Minimum rank problem with different approaches

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What is the smallest possible rank?

$$\begin{pmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

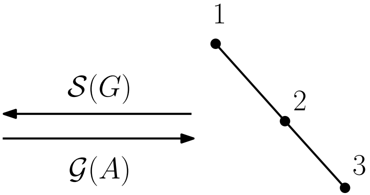
$$\begin{pmatrix} ? & * & 0 \\ * & ? & * \\ 0 & * & ? \end{pmatrix}$$

min rank (mr) $= 2$;

max nullity (M) $= 1$.

Minimum Rank / Maximum Nullity

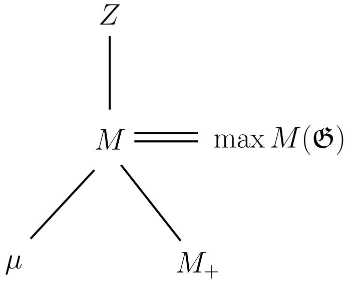
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & ? & * & 0 \\ 2 & * & ? & * \\ 3 & 0 & * & ? \end{array}$$



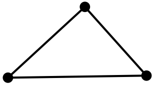
$$\begin{aligned} \operatorname{mr}(G) &= \min\{\operatorname{rank}(A) : A \in \mathcal{S}(G)\} \\ +) \quad M(G) &= \max\{\operatorname{nullity}(A) : A \in \mathcal{S}(G)\} \end{aligned}$$

$$\operatorname{mr}(G) + M(G) = \text{rank} + \text{nullity} = |V(G)|$$

All Related Parameters



Example: K_n

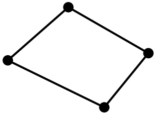


$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} ? & * & * \\ * & ? & * \\ * & * & ? \end{pmatrix}$$

$$M(K_n) = n - 1, \ n \geq 2$$

Example: C_n

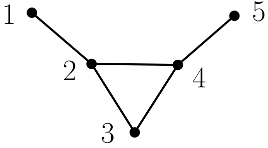


$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} ? & * & 0 & * \\ * & ? & * & 0 \\ 0 & * & ? & * \\ * & 0 & * & ? \end{pmatrix}$$

$$\begin{pmatrix} ? & * & 0 & * \\ * & ? & * & 0 \\ 0 & * & ? & * \\ * & 0 & * & ? \end{pmatrix}$$

$$M(C_n) = n - 2,\ n \geq 3$$

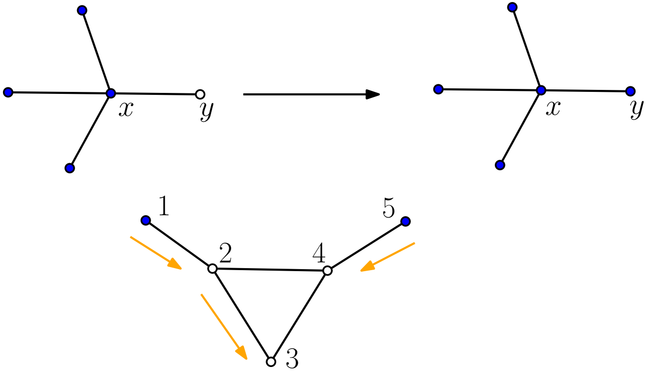


$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & ? & * & 0 & 0 & 0 \\ 2 & * & ? & * & * & 0 \\ 3 & 0 & * & ? & * & 0 \\ 4 & 0 & * & * & ? & * \\ 5 & 0 & 0 & 0 & * & ? \end{array}$$

$$\begin{array}{c}\mathrm{tri}(G) = 3 \\ \rotatebox{90}{$\leqslant$} \\ \mathrm{mr}(G) = 3\end{array}$$

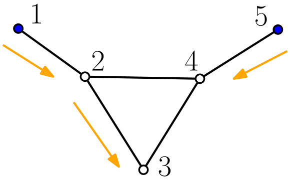
$$\begin{array}{c|ccccc} & 1 & 5 & 2 & 3 & 4 \\ \hline 2 & * & 0 & ? & * & * \\ 4 & 0 & * & * & * & ? \\ 3 & 0 & 0 & * & ? & * \\ 1 & ? & 0 & * & 0 & 0 \\ 5 & 0 & ? & 0 & 0 & * \end{array}$$

Zero Forcing Number $Z(G)$



$$Z(G) = \min |\text{initial set}| = 2$$

$Z(G)$ and $\mathrm{tri}(G)$



	1	5	2	3	4
2	$*$	0	?	$*$	$*$
4	0	$*$	$*$	$*$	?
3	0	0	$*$	?	$*$
1	?	0	$*$	0	0
5	0	?	0	0	$*$

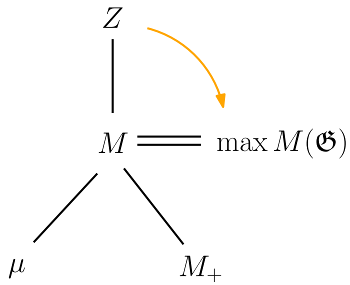
1 → 2

5 → 4

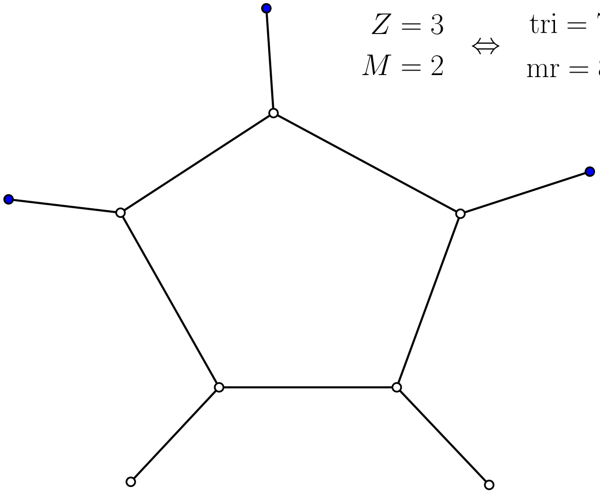
2 → 3

$$\begin{matrix} Z & + & \mathrm{tri} & = & n \\ \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\leq$} & & \\ M & + & \mathrm{mr} & = & n \end{matrix}$$

All Related Parameters



Example: $M(G) < Z(G)$



$$Z = 3 \qquad \text{tri} = 7$$
$$\Leftrightarrow$$
$$M = 2 \qquad \text{mr} = 8$$

What happened on H_5?

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{pmatrix}
0/* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & ? & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\
0 & 0 & ? & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & ? & 0 & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ? & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * & ? & 0 & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & ? & * & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & * & * & ? & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ? & * \\
0 & * & 0 & 0 & 0 & 0 & 0 & * & * & ?
\end{pmatrix} \end{array}$$

What happened on H_5?

	1	2	3	4	5	6	7	8	9	10
1	0	*	0	0	0	0	0	0	0	0
2	*	?	0	*	0	0	0	0	0	*
3	0	0	?	*	0	0	0	0	0	0
4	0	*	*	?	0	*	0	0	0	0
5	0	0	0	0	?	*	0	0	0	0
6	0	0	0	*	*	?	0	*	0	0
7	0	0	0	0	0	0	?	*	0	0
8	0	0	0	0	0	*	*	?	0	*
9	0	0	0	0	0	0	0	0	?	*
10	0	*	0	0	0	0	0	*	*	?

What happened on H_5?

	1	3	9	2	4	10	6	8	5	7
2	*	0	0	?	*	*	0	0	0	0
4	0	*	0	*	?	0	*	0	0	0
10	0	0	*	*	0	?	0	*	0	0
1	0	0	0	*	0	0	0	0	0	0
6	0	0	0	0	*	0	?	*	*	0
8	0	0	0	0	0	*	*	?	0	*
5	0	0	0	0	0	0	*	0	?	0
7	0	0	0	0	0	0	0	*	0	?
3	0	?	0	0	*	0	0	0	0	0
9	0	0	?	0	0	*	0	0	0	0

What happened on H_5?

	1	2	3	4	5	6	7	8	9	10
1	$*$	$*$	0	0	0	0	0	0	0	0
2	$*$	?	0	$*$	0	0	0	0	0	$*$
3	0	0	?	$*$	0	0	0	0	0	0
4	0	$*$	$*$	?	0	$*$	0	0	0	0
5	0	0	0	0	?	$*$	0	0	0	0
6	0	0	0	$*$	$*$	?	0	$*$	0	0
7	0	0	0	0	0	0	?	$*$	0	0
8	0	0	0	0	0	$*$	$*$	?	0	$*$
9	0	0	0	0	0	0	0	0	?	$*$
10	0	$*$	0	0	0	0	0	$*$	$*$	?

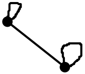
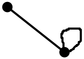
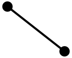
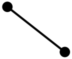
What happened on H_5?

	3	5	4	6	1	2	8	10	7	9
4	*	0	?	*	0	*	0	0	0	0
6	0	*	*	?	0	0	*	0	0	0
2	0	0	*	0	*	?	0	*	0	0
8	0	0	0	*	0	0	?	*	*	0
1	0	0	0	0	*	*	0	0	0	0
10	0	0	0	0	0	*	*	?	0	*
7	0	0	0	0	0	0	*	0	?	0
9	0	0	0	0	0	0	0	*	0	?
3	?	0	*	0	0	0	0	0	0	0
5	0	?	0	*	0	0	0	0	0	0

Minimum rank for loop graphs

Loop graph $\mathfrak{G}$

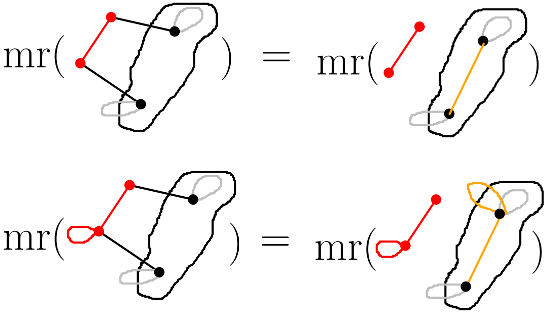
Simple graph G

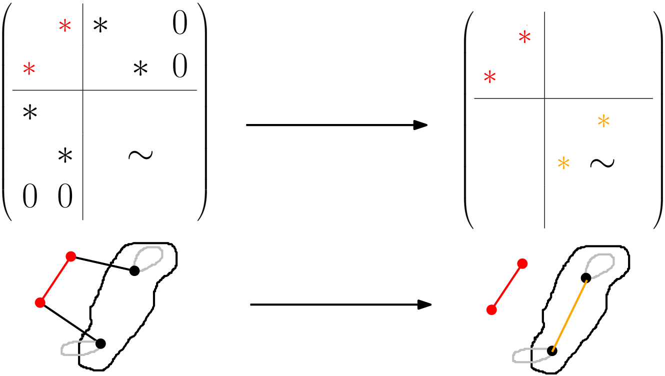
$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \qquad \begin{pmatrix} ? & * \\ * & ? \end{pmatrix}$$

$$\mathrm{mr}(\mathfrak{G}_1) = 1 \quad \mathrm{mr}(\mathfrak{G}_2) = 2 \quad \mathrm{mr}(\mathfrak{G}_3) = 2 \xrightarrow{\min} \mathrm{mr}(G) = 1$$

P_4 Reduction Lemma



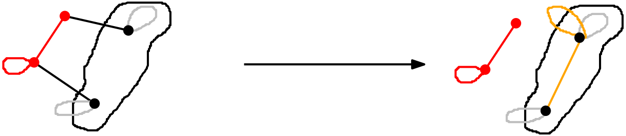
Type 1 Reduction



$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Type 2 Reduction

$$\left(\begin{array}{cc|ccc} & * & * & & 0 \\ * & * & & * & 0 \\ \hline * & & & & \\ & * & & \sim & \\ 0 & 0 & & & \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} & * & & & \\ * & * & & & \\ \hline & & *+* & * & \\ & & * & \sim & \\ & & & & \end{array}\right)$$



$$\begin{pmatrix} * & \\ & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * \\ * & \end{pmatrix} \begin{pmatrix} * & & 0 \\ & * & 0 \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

scale row 3 and row 4

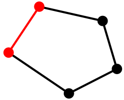
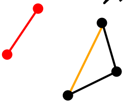
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix}$$

Schur complement

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

realized by

definition

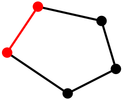
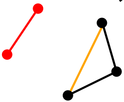
$5 = \mathrm{mr}($  $) = \mathrm{mr}($  $) = 3+2 = 5$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

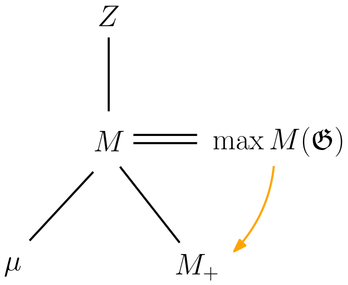
reconstruction

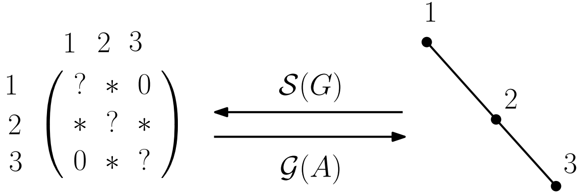
$$\begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

realized by

$5 = \mathrm{mr}($  $) = \mathrm{mr}($  $) = 3 + 2 = 5$

All Related Parameters





$$\mathcal{S}_+(G) = \{A\ p.s.d.,\ A \in \mathcal{S}(G)\} \qquad \subseteq\ \mathcal{S}(G)$$

$$M_+(G) := \max\{\text{nullity}(A) :\ A \in \mathcal{S}_+(G)\} \qquad \leq M(G)$$

PSD Decomposition

If A is p.s.d.,

$$A = Q^\top D Q = Q^\top \sqrt{D}\sqrt{D} Q = S^\top S$$

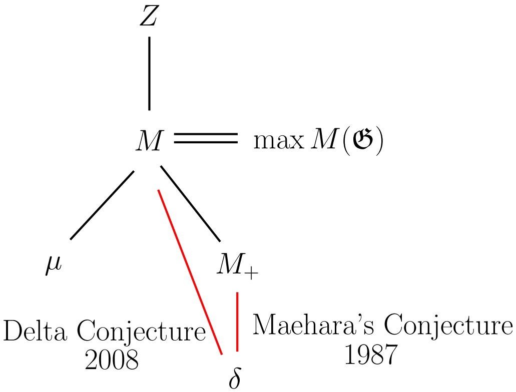
$$\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = [\langle v_i, v_j \rangle]$$

Orthtogonal Representation

$$\begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = [\langle v_i, v_j \rangle]$$

$$\begin{aligned} f : V(G) &\longrightarrow \mathbb{R}^d \\ i &\longmapsto v_i \end{aligned} \qquad \begin{aligned} \langle v_i, v_j \rangle \neq 0 &\Leftrightarrow i \sim j \\ \langle v_i, v_j \rangle = 0 &\Leftrightarrow i \nsim j \end{aligned}$$

smallest $d \Leftrightarrow$ p.s.d. minimum rank



Tracy Hall provided a proof in 2009.

$A \in \mathcal{S}(G)$

A is generalized Laplacian

A has exactly one negative eigenvalue

A satisfies Strong Arnold Hypothesis:

$$\text{No such } X \text{ exists that} \quad \begin{array}{r} AX = 0 \\ A \circ X = 0 \\ I \circ X = 0 \end{array}$$

$$\mu = \max \text{ nullity} \leq M$$

Properties of μ

$\mu(G) \leq 1 \Leftrightarrow G$ is disjoint union of paths;

$\mu(G) \leq 2 \Leftrightarrow G$ is outerplanar;

$\mu(G) \leq 3 \Leftrightarrow G$ is planar;

$\mu(G) \leq 4 \Leftrightarrow G$ is linklessly embeddable.

$Z(G) + Z(\overline{G}) \geq n - 2?$	True, by Tracy Hall.
$M(G) + M(\overline{G}) \geq n - 2?$	Unknown.
$\mu(G) + \mu(\overline{G}) \geq n - 2?$	Unknown.

$Z(G) + Z(\overline{G}) \geq n - 2$?	True, by Tracy Hall.
$M(G) + M(\overline{G}) \geq n - 2$?	Unknown.
$\mu(G) + \mu(\overline{G}) \geq n - 2$?	Unknown.

Thank you!