Generalizations of the Strong Arnold Property and the minimum number of distinct eigenvalues of a graph

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Abstract

For a given graph G and an associated class of real symmetric matrices whose off-diagonal entries are governed by the adjacencies in G, the collection of all possible spectra for such matrices is considered. Building on the pioneering work of Colin de Verdière in connection with the Strong Arnold Property, two extensions are devised that target a better understanding of all possible spectra and their associated multiplicities. These new properties are referred to as the Strong Spectral Property and

the Strong Multiplicity Property. Finally, these ideas are applied to the minimum number of distinct eigenvalues associated with G, denoted by $q(G)$. The graphs for which $q(G)$ is at least the number of vertices of G less one are characterized.

1 Introduction

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering. The general form of an inverse eigenvalue problem is the following: given a family $\mathcal{F}$ of matrices and a spectral property $\mathcal{P}$, determine if there exists a matrix $A \in \mathcal{F}$ with property $\mathcal{P}$. Examples of families are tridiagonal, Toeplitz, or all symmetric matrices with a given graph. Examples of properties include: having a prescribed rank, a prescribed spectrum, a prescribed eigenvalue and corresponding eigenvector, or a prescribed list of multiplicities. Our focus is on inverse eigenvalue problems where $\mathcal{F}$ is a set of symmetric matrices associated with a graph. These have received considerable attention, and a rich mathematical theory has been developed around them (see, for example, [13]).

All matrices in this paper are real. Let $G = (V(G), E(G))$ be a (simple, undirected) graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. The *set $\mathcal{S}(G)$ of symmetric matrices described by G* consists of the set of all symmetric $n \times n$ matrices $A = (a_{ij})$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E(G)$. We denote the spectrum of A, i.e., the multiset of eigenvalues of A, by spec(A). The *inverse spectrum problem for G*, also known as the *inverse eigenvalue problem for G*, refers to determining the possible spectra that occur among the matrices in $\mathcal{S}(G)$. The inverse spectrum problem for G seems to be difficult, as evidenced by that fact that it has been completely solved for only a few special families of graph, e.g. paths, generalized stars, double generalized stars, and complete graphs [7, 9, 21].

To gain a better understanding of the inverse eigenvalue problem for graphs, other spectral properties have been studied. For example, the *maximum multiplicity problem for G* is: Determine $\mathrm{M}(G)$, where

$$\mathrm{M}(G) = \max\{\mathrm{mult}_A(\lambda) : A \in \mathcal{S}(G), \ \lambda \in \mathrm{spec}(A)\},$$

and $\mathrm{mult}_A(\lambda)$ denotes the multiplicity of λ as an eigenvalue of A. A related invariant is the minimum rank of G, which is defined by

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}.$$

The *minimum rank problem* is: Given a graph G, determine $\mathrm{mr}(G)$. As $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where $|G|$ denotes the number of vertices of G, the maximum multiplicity and minimum rank problems are essentially the same. These problems have been extensively studied in recent years; see [13, 14] for surveys. If the distinct eigenvalues of A are $\lambda_1 < \lambda_2 < \cdots < \lambda_q$ and the multiplicity of these eigenvalues are $m_1, m_2, \ldots, m_q$ respectively, then the *ordered multiplicity list of A* is $\mathbf{m} = (m_1, m_2, \ldots, m_q)$. This notion gives rise to the *inverse ordered multiplicity list problem*: Given a graph G, determine which ordered multiplicity lists arise among the matrices in $\mathcal{S}(G)$. This problem has

been studied in [6, 7, 21]. A recently introduced spectral problem (see [1]) is the *minimum number of distinct eigenvalues problem*: Given a graph G, determine $q(G)$ where $q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}$ and $q(A)$ is the number of distinct eigenvalues of A.

For a specific graph G and a specific property $\mathcal{P}$, it is often difficult to find an explicit matrix $A \in \mathcal{S}(G)$ having property $\mathcal{P}$ (e.g., consider the challenge of finding a matrix whose graph is a path on five vertices and that has eigenvalues 0, 1, 2, 3, 4).

In this paper, we carefully describe the underlying theory of a technique based on the implicit function theorem, and develop new methods for two types of inverse eigenvalue problems. Suppose G is a graph and $\mathcal{P}$ is a spectral property (such as having a given spectrum, given ordered multiplicity list, or given multiplicity of the eigenvalue 0) of a matrix in $\mathcal{S}(G)$. The theory is applied to determine conditions (dependent on the property $\mathcal{P}$) that guarantee if a matrix $A \in \mathcal{S}(G)$ has property $\mathcal{P}$ and satisfies these conditions, then for every supergraph $\widetilde{G}$ of G (on the same vertex set) there is a matrix $B \in \mathcal{S}(\widetilde{G})$ satisfying property $\mathcal{P}$. (A graph $\widetilde{G}$ is a *supergraph* of G if G is a subgraph of $\widetilde{G}$.) In Section 2, the technique is developed, and related to both the implicit function theorem and the work of Colin de Verdière [10, 11]. The technique is then applied to produce two new properties for symmetric matrices called the Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP) that generalize the Strong Arnold Property (SAP).

In Section 3, we establish general theorems for the inverse spectrum problem, respectively the inverse multiplicity list problem, using matrices satisfying the SSP, respectively the SMP. In Section 4, we use the SSP and SMP to prove properties of $q(G)$. In particular we answer a question raised in [1] by giving a complete characterization of the graphs G for which $q(G) \geqslant |G| - 1$.

2 Motivation and fundamental results

We begin by recalling an inverse problem due to Colin de Verdière. In his study of certain Schrödinger operators, Colin de Verdière was concerned with the maximum nullity, $\mu(G)$, of matrices in the class of matrices $A \in \mathcal{S}(G)$ satisfying:

(i) all off-diagonal entries of A are non-positive;

(ii) A has a unique negative eigenvalue with multiplicity 1; and

(iii) O is the only symmetric matrix X satisfying $AX = O$, $A \circ X = O$ and $I \circ X = O$.

Here $\circ$ denotes the Schur (also known as the Hadamard or entrywise) product and O denotes the zero matrix. Condition (iii) is known as the *Strong Arnold Property* (or SAP for short). Additional variants, called *Colin de Verdière type parameters*, include $\xi(G)$, which is the maximum nullity of matrices $A \in \mathcal{S}(G)$ satisfying the SAP [8], and $\nu(G)$, which is the maximum nullity of positive semidefinite matrices $A \in \mathcal{S}(G)$ satisfying the SAP [11]. The maximum multiplicity $\mathrm{M}(G)$ is also the maximum nullity of a matrix in $\mathcal{S}(G)$, so $\xi(G) \leqslant \mathrm{M}(G)$. Analogously, $\nu(G) \leqslant \mathrm{M}_+(G)$, where $\mathrm{M}_+(G)$ is the maximum nullity among positive semidefinite matrices in $\mathcal{S}(G)$.

Colin de Verdière [10] used results from manifold theory to show conditions equivalent to (i)–(iii) imply that if G is a subgraph of $\widetilde{G}$, then the existence of $A \in \mathcal{S}(G)$ satisfying (i)–(iii), implies the existence of $\widetilde{A} \in \mathcal{S}(\widetilde{G})$ having the same nullity as A and satisfying (i)–(iii).[1] The formulation of the SAP in (iii) is due to van der Holst, Lovász and Schrijver [19], which gives a linear algebraic treatment of the SAP and the Colin de Verdière number μ. Using a technique similar to that in [19], Barioli, Fallat, and Hogben [8] showed that if there exists $A \in \mathcal{S}(G)$ satisfying the SAP and G is a subgraph of $\widetilde{G}$, then there exists $\widetilde{A} \in \mathcal{S}(\widetilde{G})$ such that $\widetilde{A}$ has the SAP and A and $\widetilde{A}$ have the same nullity.[2]

The Strong Arnold Property has been used to obtain many results about the maximum nullity of a graph. Our goal in this section is to first describe the general technique behind Colin de Verdière's work, and then develop analogs of the SAP for the inverse spectrum problem and the inverse multiplicity list problem. For convenience, we state below a version of the Implicit Function Theorem (IFT) because it is central to the technique; see [12, 22].

Theorem 1. *Let $F : \mathbb{R}^{s+r} \to \mathbb{R}^t$ be a continuously differentiable function on an open subset U of $\mathbb{R}^{s+r}$ defined by*

$$F(\mathbf{x}, \mathbf{y}) = (F_1(\mathbf{x}, \mathbf{y}), F_2(\mathbf{x}, \mathbf{y}), \ldots, F_t(\mathbf{x}, \mathbf{y})),$$

where $\mathbf{x} = (x_1, \ldots, x_s)^\top \in \mathbb{R}^s$ and $\mathbf{y} \in \mathbb{R}^r$. Let $(\mathbf{a}, \mathbf{b})$ be an element of U with $\mathbf{a} \in \mathbb{R}^s$ and $\mathbf{b} \in \mathbb{R}^r$, and $\mathbf{c} \in \mathbb{R}^t$ such that $F(\mathbf{a}, \mathbf{b}) = \mathbf{c}$. If the $t \times s$ matrix

$$\left(\frac{\partial F_i}{\partial x_j} \bigg|_{(\mathbf{a}, \mathbf{b})} \right) \tag{1}$$

has rank t, then there exist an open neighborhood V containing $\mathbf{a}$ and an open neighborhood W containing $\mathbf{b}$ such that $V \times W \subseteq U$ and a continuous function $\phi : W \to V$ such that $F(\phi(\mathbf{y}), \mathbf{y}) = \mathbf{c}$ for all $\mathbf{y} \in W$.

The IFT concerns robustness of solutions to the system $F(\mathbf{x}, \mathbf{y}) = \mathbf{c}$. Namely, the existence of a "nice" solution $\mathbf{x} = \mathbf{a}$ to $F(\mathbf{x}, \mathbf{b}) = \mathbf{c}$ guarantees the existence of solutions to all systems $F(\mathbf{x}, \tilde{\mathbf{b}}) = \mathbf{c}$ with $\tilde{\mathbf{b}}$ sufficiently close to $\mathbf{b}$. Here, "nice" means that the columns of the matrix in (1) span $\mathbb{R}^t$.

Remark 2. More quantitative proofs of the implicit function theorem (see [22, Theorem 3.4.10]) show that there exists an $\epsilon > 0$ that depends only on U and the Jacobian in (1) such that there is such a continuous function ϕ with $F(\phi(\mathbf{y}), \mathbf{y}) = \mathbf{c}$ for all $\mathbf{y}$ with $\|\mathbf{y} - \mathbf{b}\| < \epsilon$.

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[1]In fact, much more was shown: If G is a minor of H, then there exists $B \in \mathcal{S}(H)$ satisfying (i)–(iii) such that A and B have the same nullity. A *minor* of a graph is obtained by contracting edges and by deleting edges and vertices, and a graph parameter β is *minor monotone* if $\beta(G) \leqslant \beta(H)$.

[2]Again, the result was established for minors: If G is a minor of H, then there exists $B \in \mathcal{S}(H)$ satisfying the SAP such that A and B have the same nullity.

A useful application of the IFT arises in the setting of smooth manifolds. We refer the reader to [22, 23] for basic definitions and results about manifolds. Given a manifold $\mathcal{M}$ embedded smoothly in an inner product space and a point $\mathbf{x}$ in $\mathcal{M}$, we denote the tangent space to $\mathcal{M}$ at $\mathbf{x}$ by $\mathcal{T}_{\mathcal{M}.\mathbf{x}}$ and the normal space[3] to $\mathcal{M}$ at $\mathbf{x}$ by $\mathcal{N}_{\mathcal{M}.\mathbf{x}}$. For the manifolds of matrices discussed here, the inner product of $n \times n$ matrices A and B is $\langle A, B \rangle = \text{tr}(A^\top B)$, or equivalently, the Euclidean inner product on $\mathbb{R}^{n^2}$ (with matrices viewed as n^2-tuples).

Many problems, including our inverse problems, can be reduced to determining whether or not the intersection of two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ is non-empty. There is a condition known as transversality such that if $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect transversally at $\mathbf{x}$, then any manifold "near" $\mathcal{M}_1$ and any manifold "near" $\mathcal{M}_2$ intersect non-trivially. In other words, the existence of a nice solution implies the existence of a solution to all "nearby" problems. More precisely, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be manifolds in some $\mathbb{R}^d$, and $\mathbf{x}$ be a point in $\mathcal{M}_1 \cap \mathcal{M}_2$. The manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect *transversally* at $\mathbf{x}$ provided $\mathcal{T}_{\mathcal{M}_1.\mathbf{x}} + \mathcal{T}_{\mathcal{M}_2.\mathbf{x}} = \mathbb{R}^d$, or equivalently $\mathcal{N}_{\mathcal{M}_1.\mathbf{x}} \cap \mathcal{N}_{\mathcal{M}_2.\mathbf{x}} = \{O\}$.

By a smooth family $\mathcal{M}(s)$ $(s \in (-1,1))$ of manifolds in $\mathbb{R}^d$ we mean that $\mathcal{M}(s)$ is a manifold in $\mathbb{R}^d$ for each $s \in (-1,1)$, and $\mathcal{M}(s)$ varies smoothly as a function of s. Thus, if $\mathcal{M}(s)$ is a smooth family of manifolds in $\mathbb{R}^d$, then

(i) there is a k such that $\mathcal{M}(s)$ has dimension k for all $s \in (-1,1)$;

(ii) there exists a collection $\{U_\alpha : \alpha \in \mathcal{I}\}$ of relatively open sets whose union is $\cup_{s \in (-1,1)} \mathcal{M}(s)$; and

(iii) for each α there is a diffeomorphism $F_\alpha : (-1,1)^{k+1} \to U_\alpha$.

Note that if $\mathbf{x}_0 \in (-1,1)^k$, $s_0 \in (-1,1)$, $\mathbf{y}_0 \in \mathcal{M}(s_0)$ and $F_\alpha(\mathbf{x}_0, s_0) = \mathbf{y}_0$, then the tangent space to $\mathcal{M}(s_0)$ at $\mathbf{y}_0$ is the column space of the $d \times k$ matrix

$$\left(\frac{\partial F_\alpha}{\partial x_j} \Big|_{(\mathbf{x}=\mathbf{x}_0, s=s_0)} \right).$$

The following theorem can be viewed as a specialization of Lemma 2.1 and Corollary 2.2 of [19] to the case of two manifolds.

Theorem 3. *Let $\mathcal{M}_1(s)$ and $\mathcal{M}_2(t)$ be smooth families of manifolds in $\mathbb{R}^d$, and assume that $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ intersect transversally at $\mathbf{y}_0$. Then there is a neighborhood $W \subseteq \mathbb{R}^2$ of the origin and a continuous function $f : W \to \mathbb{R}^d$ such that for each $\epsilon = (\epsilon_1, \epsilon_2) \in W$, $\mathcal{M}_1(\epsilon_1)$ and $\mathcal{M}_2(\epsilon_2)$ intersect transversally at $f(\epsilon)$.*

Proof. Let k, respectively ℓ, be the common dimension of each of the manifolds $\mathcal{M}_1(s)$ $(s \in (-1,1))$ and each of the manifolds $\mathcal{M}_2(t)$ $(t \in (-1,1))$, respectively. Let $\{U_\alpha : \alpha \in \mathcal{I}\}$ and $F_\alpha : (-1,1)^{k+1} \to U_\alpha$ be the open sets and diffeomorphisms for $\mathcal{M}_1(t)$ discussed

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[3]That is, the orthogonal complement of the tangent space in the ambient inner product space.

above. Let $\{V_\beta : \beta \in \mathcal{J}\}$ and $G_\beta : (-1,1)^{\ell+1} \to V_\beta$ be the similar open sets and diffeomorphisms for $\mathcal{M}_2(t)$.

Choose α so that $\mathbf{y}_0 \in U_\alpha$ and β so that $\mathbf{y}_0 \in V_\beta$. Define

$$H : \mathbb{R}^k \times (-1,1) \times \mathbb{R}^\ell \times (-1,1) \to \mathbb{R}^d \text{ by } H(\mathbf{u}, s, \mathbf{v}, t) = F_\alpha(\mathbf{u}, s) - G_\beta(\mathbf{v}, t),$$

where $\mathbf{u} \in (-1,1)^k$, $\mathbf{v} \in (-1,1)^\ell$, and $s, t \in (-1,1)$. There exists $\mathbf{u}(0)$ and $\mathbf{v}(0)$ such that $F_\alpha(\mathbf{u}(0), 0) = \mathbf{y}_0$ and $G_\beta(\mathbf{v}(0), 0) = \mathbf{y}_0$. Hence $H(\mathbf{u}(0), 0, \mathbf{v}(0), 0) = 0$. Since F and G are diffeomorphisms, the Jacobian of the function H restricted to $s = 0$ and $t = 0$ and evaluated at $\mathbf{u} = \mathbf{u}(0)$ and $\mathbf{v} = \mathbf{v}(0)$ is the $d \times (k + \ell)$ matrix

$$\text{Jac} = \left(\left. \frac{\partial F_i}{\partial u_j} \right|_{(\mathbf{u}(0),0)} \quad \middle| \quad \left. \frac{\partial G_i}{\partial v_j} \right|_{(\mathbf{v}(0),0)} \right).$$

The assumption that $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ intersect transversally at $\mathbf{y}(0)$ implies that the column space of Jac is all of $\mathbb{R}^d$.

The result now follows by applying the implicit function theorem (Theorem 1) and the fact that every matrix in a sufficiently small neighborhood of a full rank matrix has full rank. $\qquad \square$

2.1 The Strong Arnold Property

We now use Theorem 3 to describe a tool for the inverse rank problem. This tool is described in [19] and used there and in [4, 5, 8]. We use this problem to familiarize the reader with the general technique.

Let G be a graph of order n, and $A \in \mathcal{S}(G)$. For this problem the two manifolds that concern us are $\mathcal{S}(G)$ and the manifold

$$\mathcal{R}_A = \{B \in S_n(\mathbb{R}) : \text{rank}(B) = \text{rank}(A)\},$$

consisting of all $n \times n$ symmetric matrices with the same rank as A. We view both of these as subsets of $S_n(\mathbb{R})$, the set of all $n \times n$ symmetric matrices endowed with the inner product $\langle V, W \rangle = \text{tr}(VW)$. Thus $\mathcal{S}(G)$ and $\mathcal{R}_A$ can be thought of as submanifolds of $\mathbb{R}^{n(n+1)/2}$. It is easy to see that

$$\begin{aligned} \mathcal{T}_{\mathcal{S}(G).A} &= \{X \in S_n(\mathbb{R}) : x_{ij} \neq 0 \implies ij \text{ is an edge of } G \text{ or } i = j \}, \text{ and} \\ \mathcal{N}_{\mathcal{S}(G).A} &= \{X \in S_n(\mathbb{R}) : A \circ X = O \text{ and } I \circ X = O\}. \end{aligned}$$

For $\mathcal{N}_{\mathcal{R}_A.A}$ we have the following result [19].

Lemma 4. *Let A be a symmetric $n \times n$ matrix of rank r. Then*

$$\mathcal{N}_{\mathcal{R}_A.A} = \{X \in S_n(\mathbb{R}) : AX = O\}.$$

Proof. There exists an invertible $r \times r$ principal submatrix of A, and by permutation similarity we may take this to be the leading principal submatrix. Hence A has the form

$$
\left(
\begin{array}{c|c}
A_1 & A_1 U \\
\hline
U^\top A_1 & U^\top A_1 U
\end{array}
\right)
$$

for some invertible $r \times r$ matrix A_1 and some $r \times (n-r)$ matrix U.

Let $B(t)$ ($t \in (-1,1)$) be a differentiable path of symmetric rank r matrices such that $B(0) = A$. For t sufficiently small, the leading $r \times r$ principal submatrix of $B(t)$ is invertible and $B(t)$ has the form

$$
\left(
\begin{array}{c|c}
B_1(t) & B_1(t)U(t) \\
\hline
U(t)^\top B_1(t) & U(t)^\top B_1(t)U(t)
\end{array}
\right),
$$

where $B_1(t)$ and $U(t)$ are differentiable, $B_1(0) = A_1$ and $U(0) = U$. Differentiating with respect to t and then evaluating at $t = 0$ gives

$$
\dot{B}(0) = \left(
\begin{array}{c|c}
\dot{B}_1(0) & \dot{B}_1(0)U \\
\hline
U^\top \dot{B}_1(0) & U^\top \dot{B}_1(0)U
\end{array}
\right) + \left(
\begin{array}{c|c}
O & A_1 \dot{U}(0) \\
\hline
\dot{U}(0)^\top A_1 & \dot{U}(0)^\top A_1 U + U^\top A_1 \dot{U}(0)
\end{array}
\right).
$$

It follows that $\mathcal{T}_{\mathcal{R}_A.A} = \mathcal{T}_1 + \mathcal{T}_2$, where

$$
\mathcal{T}_1 := \left\{ \left(
\begin{array}{c|c}
R & RU \\
\hline
U^\top R & U^\top RU
\end{array}
\right) : R \text{ is an arbitrary symmetric } r \times r \text{ matrix} \right\}
$$

and

$$
\mathcal{T}_2 := \left\{ \left(
\begin{array}{c|c}
O & A_1 S \\
\hline
S^\top A_1 & S^\top A_1 U + U^\top A_1 S
\end{array}
\right) : S \text{ is an arbitrary } r \times (n-r) \text{ matrix} \right\}.
$$

Consider an $n \times n$ symmetric matrix

$$
W = \left(
\begin{array}{cc}
C & D \\
D^\top & E
\end{array}
\right),
$$

where C is $r \times r$. Then $W \in \mathcal{T}_2^\perp$ if and only if

$$
\operatorname{tr}(DS^\top A_1 + D^\top A_1 S + ES^\top A_1 U + EU^\top A_1 S) = 0 \text{ for all } S
$$

or equivalently

$$
\operatorname{tr}((D^\top A_1 + EU^\top A_1)S) = 0 \text{ for all } S.
$$

Thus, $W \in \mathcal{T}_2^\perp$ if and only if $D^\top A_1 + EU^\top A_1 = O$, which is equivalent to $D^\top = -EU^\top$ since A_1 is invertible. Similarly, $W \in \mathcal{T}_1^\perp$ if and only if $C + DU^\top + UD^\top + UEU^\top$ is skew symmetric. As $C + DU^\top + UD^\top + UEU^\top$ is symmetric, we have $C \in \mathcal{T}_1^\perp$ if and only if $C + DU^\top + UD^\top + UEU^\top = O$. For $W \in \mathcal{T}_1^\perp \cap \mathcal{T}_2^\perp$, $C = UEU^\top$ and therefore

$$\mathcal{N}_{\mathcal{R}_A.A} = \left\{ \left(\begin{array}{c|c} UEU^\top & -UE \\ \hline -EU^\top & E \end{array} \right) : E \text{ is an arbitrary symmetric } (n-r) \times (n-r) \text{ matrix} \right\}.$$
(2)

It is easy to verify that this is precisely that set of symmetric matrices X such that $AX = O$. $\qquad\square$

Lemma 4 implies that $\mathcal{R}_A$ and $\mathcal{S}(G)$ intersect transversally at A if and only if A satisfies the SAP. Now that we know the pertinent tangent spaces for the inverse rank problem for G, we can apply Theorem 3 to easily obtain the following useful tool, which was previously proved and used in [19].

Theorem 5. *If $A \in \mathcal{S}(G)$ has the SAP, then every supergraph of G with the same vertex set has a realization that has the same rank as A and has the SAP.*

Proof. Let $\widetilde{G}$ be a supergraph of G with the same vertex set, and define $\Delta = E(\widetilde{G}) - E(G)$. For $t \in (-1, 1)$, define the manifold $\mathcal{M}(t)$ as those $B = (b_{ij}) \in S_n(\mathbb{R})$ with $b_{ij} \neq 0$ if $ij \in E(G)$, $b_{ij} = t$ if $ij \in \Delta$, and $b_{ij} = 0$ if $i \neq j$ and $ij \notin E(\widetilde{G})$. Since $\mathcal{M}(0) = \mathcal{S}(G)$, and A has the SAP, $\mathcal{R}_A$ and $\mathcal{M}(0)$ intersect transversally at A. Therefore Theorem 3 guarantees a continuous function f such that for ϵ sufficiently small, $\mathcal{R}_A$ and $\mathcal{M}(\epsilon)$ intersect transversally at $f(\epsilon)$, so $f(\epsilon)$ has the same rank as A and $f(\epsilon)$ has the SAP. For $\epsilon > 0$, $f(\epsilon) \in \mathcal{S}(\widetilde{G})$. $\qquad\square$

2.2 The Strong Spectral Property

We now follow the general method outlined in the previous subsection to derive an analog of the SAP and Theorem 5 for the inverse spectrum problem. Certain aspects of this section were in part motivated by discussions with Dr. Francesco Barioli in connection with the inverse eigenvalue problem for graphs [3].

Given a multiset Λ of real numbers that has cardinality n, the set of all $n \times n$ symmetric matrices with spectrum Λ is denoted by $\mathcal{E}_\Lambda$. Thus if $A \in \mathcal{E}_\Lambda$, then $\mathcal{E}_\Lambda$ is all symmetric matrices cospectral with A. It is well known that $\mathcal{E}_\Lambda$ is a manifold [2]. A comment on notation: The notation $\mathcal{E}_\Lambda$ for the constant spectrum manifold was chosen because this manifold is determined by Λ. Then a symmetric matrix A is in $\mathcal{E}_{\text{spec}(A)}$. In conformity with this, the constant rank manifold containing A should be denoted $\mathcal{R}_{\text{rank } A}$, but we follow the literature in denoting it by $\mathcal{R}_A$.

Let A be a symmetric $n \times n$ matrix. The centralizer of A is the set of all matrices that commute with A, and is denoted by $\mathcal{C}(A)$. The commutator, $AB - BA$ of two matrices is denoted by $[A, B]$. The next result is well known but we include a brief proof for completeness.

Proposition 6. *Suppose $A \in S_n(\mathbb{R})$ and $\mathbf{v}_i, i = 1, \ldots, n$ is an orthonormal basis of eigenvectors for A with $A\mathbf{v}_i = \mu_i \mathbf{v}_i$ (where the μ_i need not be distinct). Then*

$$\mathcal{C}(A) = \text{span}(\{\mathbf{v}_i \mathbf{v}_j^\top : \mu_i = \mu_j\}).$$

Proof. Let $S = \text{span}(\{\mathbf{v}_i \mathbf{v}_j^\top : \mu_i = \mu_j\})$. Clearly $S \subseteq \mathcal{C}(A)$. For the reverse inclusion, observe that $\{\mathbf{v}_i \mathbf{v}_j^\top : i = 1, \ldots, n, j = 1, \ldots, n\}$ is a basis for $\mathbb{R}^{n \times n}$. Thus any $B \in \mathcal{C}(A)$ can be expressed as $B = \sum_{i=1}^n \sum_{j=1}^n \beta_{ij} \mathbf{v}_i \mathbf{v}_j^\top$. Then

$$O = [A, B] = \sum_{i=1}^n \sum_{j=1}^n \beta_{ij}(\mu_i - \mu_j)\mathbf{v}_i \mathbf{v}_j^\top.$$

By the independence of the $\mathbf{v}_i \mathbf{v}_j^\top$, $\beta_{ij}(\mu_i - \mu_j) = 0$ for all i, j. Therefore, $\mu_i \neq \mu_j$ implies $\beta_{ij} = 0$, so $\mathcal{C}(A) \subseteq S$. $\qquad \square$

Throughout this section we assume that $\lambda_1, \ldots, \lambda_q$ are the distinct eigenvalues of A and that $A = \sum_{i=1}^q \lambda_i E_i$ is the spectral decomposition of A (i.e. the E_i are mutually orthogonal idempotents that sum to I).

Lemma 7. *Let A be a symmetric $n \times n$ matrix with $\text{spec}(A) = \Lambda$. Then $\mathcal{N}_{\mathcal{E}_\Lambda. A} = \mathcal{C}(A) \cap S_n(\mathbb{R})$.*

Proof. Consider a differentiable path $B(t)$ ($t \in (-1, 1)$) on $\mathcal{E}_\Lambda$ such that $B(0) = A$. Then $B(t)$ has spectral decomposition

$$B(t) = \sum_{i=1}^q \lambda_i F_i(t).$$

Clearly $F_i(0) = E_i$ for $i = 1, 2, \ldots, q$. As $F_i(t)$ is given by a polynomial in $B(t)$ with coefficients that depend only on the spectrum of $B(t)$, i.e. on Λ, [18, Corollary to Theorem 9 (Chapter 9)], $F_i(t)$ is a differentiable function of t. Since the $F_i(t)$ are mutually orthogonal idempotents, we have that

$$\dot{F}_i(0)E_i + E_i\dot{F}_i(0) = \dot{F}_i(0) \quad \text{and} \tag{3}$$
$$\dot{F}_i(0)E_j + E_i\dot{F}_j(0) = O \qquad i, j = 1, 2, \ldots, q \text{ and } i \neq j. \tag{4}$$

Post-multiplying (3) by E_j gives

$$E_i\dot{F}_i(0)E_j = \dot{F}_i(0)E_j \qquad \text{for all } j \neq i \tag{5}$$

Pre-multiplying (4) by E_j gives

$$E_j\dot{F}_i(0)E_j = O \qquad \text{for all } j \neq i. \tag{6}$$

Post-multiplying (3) by E_i gives

$$E_i\dot{F}_i(0)E_i = O \qquad \text{for all } i. \tag{7}$$

Equation (5) implies that the image of $\dot{F}_i(0)E_j$ is contained in the image of E_i ($j \neq i$).

Consider eigenvectors $\mathbf{x}$ and $\mathbf{y}$ of A. First suppose that they correspond to the same eigenvalue, say λ_i. Since $\mathbf{x} = E_i\mathbf{x}$ and $\mathbf{y} = E_i\mathbf{y}$, (6) and (7) imply that $\mathbf{y}^\top \dot{F}_j(0)\mathbf{x} = 0$ for all j. Thus

$$
\begin{aligned}
\mathrm{tr}\left(\dot{B}(0)(\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top)\right) &= \mathrm{tr}\left(\sum_{j=1}^{q} \lambda_j \dot{F}_j(0)(\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top)\right) \\
&= \sum_{j=1}^{q} \lambda_j 2\mathbf{y}^\top \dot{F}_j(0)\mathbf{x} \\
&= 0.
\end{aligned}
$$

Therefore $\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top \in \mathcal{N}_{\mathcal{E}_\Lambda.A}$ for all such choices of $\mathbf{x}$ and $\mathbf{y}$ corresponding to the same eigenvalue.

Now suppose that $\mathbf{x}$ and $\mathbf{y}$ are unit eigenvectors of A corresponding to distinct eigenvalues, say λ_1 and λ_2. Let $\mu_3, \ldots, \mu_n$ be the remaining eigenvalues of A (with no assumption that they are distinct from each other or λ_1 or λ_2) and $\mathbf{z}_3, \ldots, \mathbf{z}_n$ a corresponding orthonormal set of eigenvectors. Let

$$
D(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \oplus \mathrm{diag}\,(\mu_3, \ldots, \mu_n),
$$

let $S = \begin{pmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z}_3 & \cdots & \mathbf{z}_n \end{pmatrix}$, and let $B(t) = SD(t)S^\top$. Then $B(t)$ is cospectral with A, $B(0) = A$, and

$$
\dot{B}(0) = S\dot{D}(0)S^\top = \begin{pmatrix} \mathbf{x} & \mathbf{y} \end{pmatrix} \begin{pmatrix} 0 & \lambda_2 - \lambda_1 \\ \lambda_2 - \lambda_1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}^\top \\ \mathbf{y}^\top \end{pmatrix} = (\lambda_2 - \lambda_1)(\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top).
$$

It follows that $\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top \in \mathcal{T}_{\mathcal{E}_\Lambda.A}$ for any eigenvectors $\mathbf{x}$ and $\mathbf{y}$ corresponding to distinct eigenvalues of A.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of eigenvectors of A. Then $\{\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top : i \leqslant j\}$ forms a basis for $S_n(\mathbb{R})$. We have shown that if $\mathbf{v}_i$ and $\mathbf{v}_j$ correspond to distinct eigenvalues, then $\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top \in \mathcal{T}_{\mathcal{E}_\Lambda.A}$ and if $\mathbf{v}_i$ and $\mathbf{v}_j$ correspond to the same eigenvalues, then $\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top \in \mathcal{N}_{\mathcal{E}_\Lambda.A}$. It follows that:

$$
\begin{aligned}
\mathcal{T}_{\mathcal{E}_\Lambda.A} &= \mathrm{span}(\{\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top : \mathbf{v}_i, \mathbf{v}_j \text{ correspond to distinct eigenvalues of } A\}). \quad (8) \\
\mathcal{N}_{\mathcal{E}_\Lambda.A} &= \mathrm{span}(\{\mathbf{v}_i\mathbf{v}_j^\top + \mathbf{v}_j\mathbf{v}_i^\top : \mathbf{v}_i, \mathbf{v}_j \text{ correspond to the same eigenvalue of } A\}). \quad (9)
\end{aligned}
$$

By Proposition 6, $\mathcal{C}(A)$ is the span of the set of matrices of the form $\mathbf{v}_i\mathbf{v}_j^\top$ where $\mathbf{v}_i$ and $\mathbf{v}_j$ correspond to the same eigenvalue of A. Thus $\mathcal{N}_{\mathcal{E}_\Lambda.A} = \mathcal{C}(A) \cap S_n(\mathbb{R})$. $\qquad \square$

Definition 8. The symmetric matrix A has the *Strong Spectral Property* (or A has the SSP for short) if the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$ and $[A, X] = O$ is $X = O$.

Observation 9. *For symmetric matrices A and X, $[A, X] = O$ if and only if AX is symmetric, so the SSP could have been defined as: The only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$ and AX is symmetric is $X = O$.*

Lemma 7 asserts that A has the SSP if and only if the manifolds $\mathcal{S}(G)$ and $\mathcal{E}_\Lambda$ intersect transversally at A, where G is the graph such that $A \in \mathcal{S}(G)$ and $\Lambda = \operatorname{spec}(A)$. A proof similar to that of Theorem 5 yields the next result.

Theorem 10. *If $A \in \mathcal{S}(G)$ has the SSP, then every supergraph of G with the same vertex set has a realization that has the same spectrum as A and has the SSP.*

For every $A \in \mathcal{S}(K_n)$, $A \circ X = O$ and $I \circ X = O$ imply $X = O$, so trivially A has the SSP. Next we discuss some additional examples.

Example 11. Let
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$
Then $\operatorname{spec}(A) = \{0, 0, \sqrt{3}, -\sqrt{3}\}$. To show that A has the SSP, consider $X \in S_4(\mathbb{R})$ such that $A \circ X = 0$, $I \circ X = O$ and $[A, X] = O$. Then X is a matrix of the form
$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & u & v \\ 0 & u & 0 & w \\ 0 & v & w & 0 \end{pmatrix}.$$

The fact that X commutes with A implies that X has all row sums and column sums equal to zero, which in turn implies $X = O$. Thus, A has the SSP, and by Theorem 10, every supergraph of $K_{1,3}$ has a realization with spectrum $\{0, 0, \sqrt{3}, -\sqrt{3}\}$.

Example 12. Let G be the star on $n \geqslant 5$ vertices having 1 as the central vertex, let $A \in \mathcal{S}(G)$, and let λ be an eigenvalue of A of multiplicity at least 3. From a theorem of Parter and Wiener (see, for example, [13, Theorem 2.1]), λ occurs on the diagonal of $A(1)$ at least 4 times.[4] Without loss of generality, we may assume that $a_{22} = a_{33} = a_{44} = a_{55} = \lambda$. Let $\mathbf{u} = [0, 0, 0, a_{15}, -a_{14}, 0, \ldots, 0]^\top$, $\mathbf{v} = [0, a_{13}, -a_{12}, 0, 0, 0, \ldots, 0]^\top$, and $X = \mathbf{u}\mathbf{v}^\top + \mathbf{v}\mathbf{u}^\top$. It can be verified that $AX = \lambda X = XA$, $A \circ X = O$ and $I \circ X = O$. Thus, A does not have the SSP. Therefore, no matrix in $\mathcal{S}(G)$ with an eigenvalue of multiplicity at least 3 has the SSP.

Example 13. Let G be as in the previous example. Let $A \in \mathcal{S}(G)$ such that no eigenvalue of A has multiplicity 3 or more. Without loss of generality we may assume that $A(1) = \oplus_{j=1}^k \lambda_j I_{n_j}$ for some distinct $\lambda_1, \ldots, \lambda_k$ and positive integers $n_1, \ldots, n_k$ with $n - 1 = n_1 +$

---

[4] $A(1)$ is the principal submatrix of A obtained by deleting row and column 1.

$n_2 + \cdots + n_k$. As every eigenvalue of A has multiplicity 2 or less, each $n_j \leqslant 3$ for $j = 1, 2, \ldots, k$.

Let X be a symmetric matrix with $[A, X] = O$, $A \circ X = O$ and $I \circ X = O$. The last two conditions imply that all entries in the first row, the first column, and on the diagonal of X are zero. This and the first condition imply that $X(1)$ is in $\mathcal{C}(A(1))$. By the distinctness of the λ_j we conclude that $X(1) = \oplus_{j=1}^{k} X_j$ where X_j is a symmetric matrix of order n_j with zeros on the diagonal. Partition A as $A = \begin{pmatrix} \alpha & \mathbf{a}^\top \\ \mathbf{a} & A(1) \end{pmatrix}$ and the partition $\mathbf{a} = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_k^\top)^\top$ conformally with $X(1)$. Then $[A, X] = O$ implies that $X_j \mathbf{a}_j = \mathbf{0}$. As $n_j \leqslant 3$, X_j is symmetric and has zeros on its diagonal and every entry of $\mathbf{a}_j$ is nonzero, this implies $X_j = O$. Thus, $X = O$ and we conclude that A has the SSP.

Observation 14. *If the diagonal matrix $D = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ has a repeated eigenvalue, say $\mu_i = \mu_j$, then D does not have the SSP as validated by the matrix X with a 1 in positions (i, j) and (j, i) and zeros elsewhere.*

Remark 15. Note that a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with distinct eigenvalues has the SSP, because $DX = XD$ implies all off-diagonal entries of X are zero. Therefore, by Theorem 10, every graph on n vertices has a realization that is cospectral with D and has the SSP. The existence of a cospectral matrix was proved in [24] via a different method.

However, not every matrix with all eigenvalues distinct has the SSP.

Example 16. Let

$$A = \begin{pmatrix} 3 & -2 & 0 & 0 & 1 \\ -2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

Then A does not have the SSP because $A \circ X = O$, $I \circ X = O$, $[A, X] = O$, but $X \neq O$. Note that $\mathrm{spec}(A) = \left\{ 2 \left(1 + \sqrt{2} \right), 4, \frac{1}{2} \left(1 + \sqrt{17} \right), 2 \left(1 - \sqrt{2} \right), \frac{1}{2} \left(1 - \sqrt{17} \right) \right\}$.

2.3 The Strong Multiplicity Property

Let $\mathbf{m} = (m_1, \ldots, m_q)$ be an ordered list of positive integers with $m_1 + m_2 + \cdots + m_q = n$. We let $\mathcal{U}_{\mathbf{m}}$ denote the set of all symmetric matrices whose ordered multiplicity list is $\mathbf{m}$. Thus, if A has multiplicity list $\mathbf{m}$, then

$$\mathcal{U}_{\mathbf{m}} = \{ B \in S_n(\mathbb{R}) : B \text{ has the same ordered multiplicity list as } A \}.$$

It follows from results in [2] that $\mathcal{U}_{\mathbf{m}}$ is a manifold. In the next lemma we determine $\mathcal{N}_{\mathcal{U}_{\mathbf{m}}, A}$.

Lemma 17. *Let A be an $n \times n$ symmetric matrix with exactly q distinct eigenvalues, spectrum Λ, and ordered multiplicity list $\mathbf{m}$. Then*

$$\mathcal{N}_{\mathcal{U}_{\mathbf{m}}.A} = \{X \in \mathcal{C}(A) \cap S_n(\mathbb{R}) : \operatorname{tr}(A^i X) = 0 \text{ for } i = 0, \ldots, q-1\}.$$

Proof. Let A be an $n \times n$ symmetric matrix with spectrum given by the multiset Λ, and let $B(t)$ ($t \in (-1, 1)$) be a differentiable path of matrices having the same ordered multiplicity list as A. Let A and $B(t)$ have spectral decomposition $A = \sum_{j=1}^{q} \lambda_j E_j$, and $B(t) = \sum_{j=1}^{q} \lambda_j(t) F_j(t)$, respectively. Then $\dot{B}(0) = \sum_{j=1}^{q} \dot{\lambda}_j(0) E_j + \sum_{j=1}^{q} \lambda_j \dot{F}_j(0)$, and we conclude that the tangent space of $\mathcal{U}_{\mathbf{m}}$ is the sum of

$$\mathcal{T} := \left\{ \sum_{j=1}^{q} c_j E_j : c_j \in \mathbb{R} \right\}$$

and the tangent space, $\mathcal{T}_{\mathcal{E}_\Lambda . A}$. Therefore

$$\mathcal{N}_{\mathcal{U}_{\mathbf{m}}.A} = \mathcal{N}_{\mathcal{E}_\Lambda . A} \cap \{S \in S_n(\mathbb{R}) : \operatorname{tr}(E_j S) = 0 \text{ for all } j = 1, \ldots, q\}. \tag{10}$$

As $\operatorname{span}(E_1, \ldots, E_q) = \operatorname{span}(I = A^0, \ldots, A^{q-1})$, $\operatorname{tr}(A^i S) = 0$ for $i = 0, \ldots, q-1$ if and only if $\operatorname{tr}(E_j S) = 0$ for $j = 1, \ldots, q$. The result now follows. $\square$

Definition 18. The $n \times n$ symmetric matrix A satisfies the *Strong Multiplicity Property* (or A has the SMP for short) provided the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$, $[A, X] = O$ and $\operatorname{tr}(A^i X) = 0$ for $i = 0, \ldots, n-1$ is $X = O$.

Remark 19. The minimal polynomial provides a dependency relation among the powers of A, so we can replace $i = 0, \ldots, n-1$ by $i = 0, \ldots, q-1$ where q is the number of distinct eigenvalues of A. Since $I \circ X = O$ and $A \circ X = O$ imply $\operatorname{tr}(IX) = 0$ and $\operatorname{tr}(AX) = 0$, we can replace $i = 0, \ldots, q-1$ by $i = 2, \ldots, q-1$. Therefore, for multiplicity lists with only 2 distinct eigenvalues, the SSP and the SMP are equivalent.

Lemma 17 asserts that $\mathcal{S}(G)$ and $\mathcal{U}_{\mathbf{m}}$ intersect transversally at A if and only if A has the SMP. A proof similar to that of Theorem 5 yields the next result.

Theorem 20. *If $A \in \mathcal{S}(G)$ has the SMP, then every supergraph of G with the same vertex set has a realization that has the same ordered multiplicity list as A and has the SMP.*

Observation 21. *Clearly the SSP implies the SMP.*

Example 16 and the next remark show the SSP and the SMP are distinct.

Remark 22. In contrast to Example 16, every symmetric matrix whose eigenvalues are distinct has the SMP. To see this, let $A \in S_n(\mathbb{R})$ have n distinct eigenvalues and let its spectral decomposition be

$$A = \sum_{i=1}^{n} \lambda_j \mathbf{y}_j \mathbf{y}_j^\top.$$

Suppose $X \in S_n(\mathbb{R})$, $I \circ X = O$, $A \circ X = O$, $[A, X] = O$, and $\operatorname{tr}(A^i X) = 0$ for $i = 0, 1, \ldots, n-1$. Since A and X commute, each eigenspace of A is an invariant subspace of X. These conditions and the distinctness of eigenvalues imply that each $\mathbf{y}_j$ is an eigenvector of X, and $\operatorname{tr}(p(A)X) = 0$ for all polynomials $p(x)$. The distinctness of eigenvalues implies that $\mathbf{y}_j \mathbf{y}_j^\top$ is a polynomial in A for $j = 1, \ldots, n$. Hence $0 = \operatorname{tr}(\mathbf{y}_j \mathbf{y}_j^\top X) = \mathbf{y}_j^\top X \mathbf{y}_j$, so the eigenvalue of X for which $\mathbf{y}_j$ is an eigenvector of X is 0 for $j = 1, \ldots, n$. Thus $X = O$, and we conclude that A has the SMP.

Observation 23. *Clearly the SMP implies the SAP.*

The next example shows that the SMP and the SAP are distinct.

Example 24. Consider the matrices

$$
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 3 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 3 & -1 & 0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 3
\end{pmatrix}, \quad
X = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

Clearly $A \circ X = O$ and $I \circ X = O$. It is straightforward to verify (using computational software) that $\operatorname{spec}(A) = \{0^{(4)}, 4^{(4)}\}$ and $[A, X] = O$, where $\lambda^{(m)}$ denotes that eigenvalue λ has multiplicity m. Since A has only two eigenvalues, A does not have the SMP by Remark 19. It is also straightforward to verify (using computational software) that both A and $A - 4I$ have the SAP. Note that $\mathcal{G}(A) = Q_3$ and A is a diagonal scaling of the positive semidefinite matrix of rank four constructed in [25, Example 2.1].

Observation 25. *If A has the SSP (SMP), then $A + \lambda I$ has the SSP (SMP) for all $\lambda \in \mathbb{R}$.*

Remark 26. If λ is the only multiple eigenvalue of A, then A has the SMP if and only if $A - \lambda I$ has the SAP. To see this assume that λ_1 is the only multiple eigenvalue of A, $A - \lambda_1 I$ has the SAP, $\lambda_2, \ldots, \lambda_q$ are the remaining eigenvalues of A, and $\mathbf{y}_j$ is a unit eigenvector of A corresponding to λ_j $(j = 2, \ldots, q)$. Then the spectral decomposition of A has the form $A = \lambda_1 E_1 + \sum_{j=2}^{q} \lambda_j \mathbf{y}_j \mathbf{y}_j^\top$. Assume X is a symmetric matrix such that $A \circ X = O$, $I \circ X = O$, $[A, X] = O$, and $\operatorname{tr}(A^k X) = 0$ for all k. As in Remark 22, each $\mathbf{y}_j$ is an eigenvector of X and $\mathbf{y}_j \mathbf{y}_j^\top$ is a polynomial in A, so $0 = \operatorname{tr}(\mathbf{y}_j \mathbf{y}_j^\top X) = \mathbf{y}_j^\top X \mathbf{y}_j$. Thus we conclude that $\mathbf{y}_j$ is in the null space of X for $j = 2, \ldots, q$. Therefore, $AX = \lambda_1 E_1 X = \lambda_1 X$ (with the latter equality coming from $E_1 + \sum_{j=2}^{q} \mathbf{y}_j \mathbf{y}_j^\top = I$). Thus, $(A - \lambda_1 I)X = O$. Since $A - \lambda_1 I$ has the SAP, $X = O$ and we conclude that A has the SMP.

3 Properties of matrices having the SSP or SMP

Section 3.1 presents characterizations of the tangent spaces $\mathcal{T}_{\mathcal{R}_A \cdot A}$, $\mathcal{T}_{\mathcal{E}_\Lambda \cdot A}$, and $\mathcal{T}_{\mathcal{U}_{\mathbf{m}} \cdot A}$ and applies these to obtain lower bounds on the number of edges in a graph where a matrix has

the associated strong property. Section 3.2 describes a computational test for determining whether a matrix has the SSP or the SMP. Section 3.3 presents the Gershgorin intersection graph and uses it to test for the SSP. Section 3.4 characterizes when block diagonal matrices have the SSP or the SMP in terms of the diagonal blocks.

3.1 Tangent spaces for the strong property manifolds

We begin by giving equivalent, but more useful, descriptions of the tangent spaces $\mathcal{T}_{\mathcal{R}_A.A}$, $\mathcal{T}_{\mathcal{E}_\Lambda.A}$, and $\mathcal{T}_{\mathcal{U}_\mathbf{m}.A}$. The set of all $n \times n$ skew-symmetric matrices is denoted by $K_n(\mathbb{R})$.

Theorem 27. *Let A be an $n \times n$ symmetric matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_q$, spectral decomposition $A = \sum_{j=1}^q \lambda_j E_j$, ordered multiplicity list $\mathbf{m} = (m_1, \ldots, m_q)$ and rank r; the spectrum of A is $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_q^{(m_q)}\}$. Then*

(a) [19] $\mathcal{T}_{\mathcal{R}_A.A} = \{AY + Y^\top A : Y \text{ is an } n \times n \text{ matrix}\}$ *and* $\dim \mathcal{T}_{\mathcal{R}_A.A} = \binom{n+1}{2} - \binom{n-r+1}{2} = \binom{r+1}{2} + r(n-r)$;

(b) $\mathcal{T}_{\mathcal{E}_\Lambda.A} = \{AK - KA : K \in K_n(\mathbb{R})\}$ *and* $\dim \mathcal{T}_{\mathcal{E}_\Lambda.A} = \binom{n}{2} - \sum_{j=1}^q \binom{m_i}{2}$;

(c) $\mathcal{T}_{\mathcal{U}_\mathbf{m}.A} = \mathcal{T}_{\mathcal{E}_\Lambda.A} + \mathrm{span}\{I = A^0, \ldots, A^{q-1}\}$ *and* $\dim \mathcal{T}_{\mathcal{U}_\mathbf{m}.A} = \binom{n}{2} - \sum_{j=1}^q \binom{m_i}{2} + q$.

Proof. Throughout the proof $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of eigenvectors of A with corresponding eigenvalues $\mu_1, \ldots, \mu_n$ with $\mu_i \neq 0$ for $i = 1, \ldots, r$ and $\mu_i = 0$ for $i = r+1, \ldots, n$.

To prove (a), consider an $n \times n$ matrix Y and $X \in \mathcal{N}_{\mathcal{R}_A.A}$. First note that Lemma 4 asserts that $X \in S_n(\mathbb{R})$ and $AX = O$. Thus $XA = O$ and

$$
\begin{aligned}
\mathrm{tr}((AY + Y^\top A)X) &= \mathrm{tr}(AYX + Y^\top AX) \\
&= \mathrm{tr}(XAY + Y^\top AX) \\
&= \mathrm{tr}(O + O) \\
&= 0.
\end{aligned}
$$

Therefore $\{AY + Y^\top A\} \subseteq \mathcal{N}_{\mathcal{R}_A.A}^\perp = \mathcal{T}_{\mathcal{R}_A.A}$. Next note that if $i \leqslant r$, then $\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top = AY + Y^\top A$ where $Y = \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_j^\top$. Define Ω to be the set of pairs (i,j) with $1 \leqslant i \leqslant j \leqslant n$ and $i \leqslant r$. Thus

$$
\begin{aligned}
\mathrm{span}\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : (i,j) \in \Omega\} &\subseteq \{AY + Y^\top A : Y \text{ is an } n \times n \text{ matrix}\} \\
&\subseteq \mathcal{T}_{\mathcal{R}_A.A}.
\end{aligned} \tag{11}
$$

It is easy to verify that $\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : (i,j) \in \Omega\}$ is an orthogonal set and hence

$$
\dim \mathrm{span}\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : (i,j) \in \Omega\} = \binom{r+1}{2} + r(n-r).
$$

By (2), $\dim \mathcal{N}_{\mathcal{R}_A.A} = \binom{n-r+1}{2}$, and hence $\dim \mathcal{T}_{\mathcal{R}_A.A} = \binom{n+1}{2} - \binom{n-r+1}{2}$. It follows that $\dim \mathrm{span}\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : (i,j) \in \Omega\} = \dim \mathcal{T}_{\mathcal{R}_A.A}$. Therefore, equality holds throughout (11), and (a) holds.

To prove (b) consider $K \in K_n(\mathbb{R})$ and $B \in \mathcal{N}_{\mathcal{E}_\Lambda . A}$. Note that Lemma 7 asserts that $B \in \mathcal{C}(A) \cap S_n(\mathbb{R})$. Thus

$$
\begin{aligned}
\operatorname{tr}((AK - KA)B) &= \operatorname{tr}(AKB - KAB) \\
&= \operatorname{tr}(K(BA - AB)) \\
&= \operatorname{tr}(O) \\
&= 0.
\end{aligned}
$$

Observe that if $\mu_i \neq \mu_j$, then $\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top = AK - KA$ where K is the skew-symmetric matrix $\frac{1}{\mu_i - \mu_j}(\mathbf{v}_i \mathbf{v}_j^\top - \mathbf{v}_j \mathbf{v}_i^\top)$. Hence,

$$
\begin{aligned}
\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : \mu_i \neq \mu_j\} &\subseteq \{AK - KA : K \in K_n(\mathbb{R})\} \\
&\subseteq \mathcal{N}_{\mathcal{E}_\Lambda . A}^\perp \\
&= \mathcal{T}_{\mathcal{E}_\Lambda . A} \\
&= \{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : \mu_i \neq \mu_j\}.
\end{aligned}
\tag{12}
$$

where the last equality is (8). Thus, equality holds throughout (12). The dimension of $\operatorname{span}\{\mathbf{v}_i \mathbf{v}_j^\top + \mathbf{v}_j \mathbf{v}_i^\top : \mu_i \neq \mu_j\}$ is easily seen to be $\binom{n+1}{2} - \sum_{j=1}^q \binom{m_i+1}{2} = \binom{n}{2} - \sum_{j=1}^q \binom{m_i}{2}$, and we have proven (b).

Statement (c) follows from (b) and (10), and the facts that

$$
\operatorname{span}\{E_1, \ldots, E_q\} = \operatorname{span}\{I = A^0, \ldots, A^{q-1}\}
$$

and

$$
\mathcal{T}_{\mathcal{E}_\Lambda . A} \cap \operatorname{span}\{A^0, \ldots, A^{q-1}\} = \{O\}
$$

since $\operatorname{span}\{A^0, \ldots, A^{q-1}\} \subseteq \mathcal{C}(A) \cap S_n(\mathbb{R}) = \mathcal{N}_{\mathcal{E}_\Lambda . A}$. $\qquad \square$

Remark 28. Let $\mathcal{M}$ be a manifold in $S_n(\mathbb{R})$ and G be a graph of order n such that $\mathcal{M}$ and $\mathcal{S}(G)$ intersect transversally at A. Then

$$
\dim \mathcal{T}_{\mathcal{M}.A} + \dim \mathcal{T}_{\mathcal{S}(G).A} - \dim (\mathcal{T}_{\mathcal{M}.A} \cap \mathcal{T}_{\mathcal{S}(G).A}) = \dim S_n(\mathbb{R}) = \binom{n+1}{2}.
$$

Since $\dim \mathcal{T}_{\mathcal{S}(G).A} = n + |E(G)|$ and $\binom{n+1}{2} = \binom{n}{2} + n$,

$$
|E(G)| = \binom{n}{2} - \dim \mathcal{T}_{\mathcal{M}.A} + \dim(\mathcal{T}_{\mathcal{M}.A} \cap \mathcal{T}_{\mathcal{S}(G).A}).
$$

Corollary 29. *Let G be a graph on n vertices and let $A \in \mathcal{S}(G)$ with spectrum Λ, ordered multiplicity list $\mathbf{m} = (m_1, \ldots, m_q)$, and rank r. Assume that A is not a scalar matrix. Then*

(a) [17] *If $\mathcal{R}_A$ and $\mathcal{S}(G)$ intersect transversally at A, then*

$$
|E(G)| \geqslant \begin{cases} \binom{n-r+1}{2} & \text{if } G \text{ is not bipartite}, \\ \binom{n-r+1}{2} - 1 & \text{if } G \text{ is bipartite}; \end{cases}
$$

(b) *If $\mathcal{E}_\Lambda$ and $\mathcal{S}(G)$ intersect transversally at A, then $|E(G)| \geqslant \sum_{j=1}^{q} \binom{m_i}{2}$; and*

(c) *If $\mathcal{U}_\mathbf{m}$ and $\mathcal{S}(G)$ intersect transversally at A, then $|E(G)| \geqslant \sum_{j=1}^{q} \binom{m_i}{2} - q + 2$.*

Proof. Statement (b) follows immediately from Remark 28 and part (b) of Theorem 27.

Statement (c) follows from Remark 28, part (c) of Theorem 27, and the fact that I and A are linearly independent, and lie in $\mathcal{T}_{\mathcal{U}_{\mathbf{m}.A}} \cap \mathcal{T}_{\mathcal{S}(G)}$.

Statement (a) can be established using Remark 28 and part (a) of Theorem 27 but the argument is more complicated. Since it was previously established in [17] (see Theorem 6.5 and Corollary 6.6) we refer the reader there. □

3.2 Equivalent criteria for the strong properties

Let H be a graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{e_1, \ldots, e_p\}$, where $e_k = i_k j_k$. For $A = (a_{i,j}) \in S_n(\mathbb{R})$, we denote the $p \times 1$ vector whose k-th coordinate is a_{i_k, j_k} by $\mathrm{vec}_H(A)$. Thus, $\mathrm{vec}_H(A)$ makes a vector out of the elements of A corresponding to the edges in H. Note that $\mathrm{vec}_H(\cdot)$ defines a linear transformation from $S_n(\mathbb{R})$ to $\mathbb{R}^p$. The *complement* $\overline{G}$ of G is the graph with the same vertex set as G and edges exactly where G does not have edges.

Proposition 30. *Let $\mathcal{M}$ be a manifold in $S_n(\mathbb{R})$, let G be a graph with vertices $1, 2, \ldots, n$ such that $A \in \mathcal{M} \cap \mathcal{S}(G)$, and let p be the number of edges in $\overline{G}$. Then $\mathcal{M}$ and $\mathcal{S}(G)$ intersect transversally at A if and only if $\{\mathrm{vec}_{\overline{G}}(B) : B \in \mathcal{T}_{\mathcal{M}.A}\} = \mathbb{R}^p$.*

Proof. First assume that $\mathcal{M}$ and $\mathcal{S}(G)$ intersect transversally at A. Consider $\mathbf{c} \in \mathbb{R}^p$, and let $C = (c_{ij}) \in S_n(\mathbb{R})$ with $c_{ij} = c_k$ if ij is the k-th edge of $\overline{G}$. The assumption of transversality implies there exist $B \in \mathcal{T}_{\mathcal{M}.A}$ and $D \in \mathcal{T}_{\mathcal{S}(G).A}$ such that $C = B + D$. Hence

$$\mathbf{c} = \mathrm{vec}_{\overline{G}}(C) = \mathrm{vec}_{\overline{G}}(B) + \mathrm{vec}_{\overline{G}}(D) = \mathrm{vec}_{\overline{G}}(B),$$

and we conclude that $\{\mathrm{vec}_{\overline{G}}(B) : B \in \mathcal{T}_{\mathcal{M}.A}\} = \mathbb{R}^p$.

Conversely, assume that $\{\mathrm{vec}_{\overline{G}}(B) : B \in \mathcal{T}_{\mathcal{M}.A}\} = \mathbb{R}^p$ and consider $C \in S_n(\mathbb{R})$. Then $\mathrm{vec}_{\overline{G}}(C) = \mathrm{vec}_{\overline{G}}(B)$ for some $B \in \mathcal{T}_{\mathcal{M}.A}$. Note that $C - B \in \mathcal{T}_{\mathcal{S}(G).A}$. Since $C = B + (C - B)$, we conclude that $\mathcal{T}_{\mathcal{M}.A} + \mathcal{T}_{\mathcal{S}(G).A} = S_n(\mathbb{R})$. □

In the following, E_{ij} denotes the $n \times n$ matrix with a 1 in position (i, j) and 0 elsewhere, and K_{ij} denotes the $n \times n$ skew-symmetric matrix $E_{ij} - E_{ji}$. Theorem 27 and Proposition 30 imply the next result.

Theorem 31. *Let G be a graph, let $A \in \mathcal{S}(G)$ with q distinct eigenvalues, and let p be the number of edges in $\overline{G}$. Then*

(a) *A has the SAP if and only if the matrix whose columns are $\mathrm{vec}_{\overline{G}}(AE_{ij} + E_{ij}^\top A)$ for $1 \leqslant i, j \leqslant n$ has rank p;*

(b) *A has the SSP if and only if the matrix whose columns are* $\mathrm{vec}_{\overline{G}}(AK_{ij} - K_{ij}A)$ *for* $1 \leqslant i < j \leqslant n$ *has rank* p; *and*

(c) *A has the SMP if and only if the matrix whose columns are* $\mathrm{vec}_{\overline{G}}(AK_{ij} - K_{ij}A)$ *for* $1 \leqslant i < j \leqslant n$ *along with* $\mathrm{vec}_{\overline{G}}(A^k)$ $(k = 0, 1, \ldots, q-1)$ *has rank* p.

Example 32. Let

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \sqrt{6} \\ 1 & 1 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 4 & 1 & 5 \\ 0 & 0 & 1 & 4 & 5 \\ \sqrt{6} & \sqrt{6} & 5 & 5 & 16 \end{pmatrix}.$$

Then

$$[A, K_{1,3}] = \begin{pmatrix} 0 & 0 & -3 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 & \sqrt{6} \\ -1 & 0 & 0 & 0 & 0 \\ -5 & 0 & \sqrt{6} & 0 & 0 \end{pmatrix}, \qquad [A, K_{1,4}] = \begin{pmatrix} 0 & 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & \sqrt{6} \\ -5 & 0 & 0 & \sqrt{6} & 0 \end{pmatrix},$$

$$[A, K_{2,3}] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -1 & -5 \\ 1 & -3 & 0 & 0 & \sqrt{6} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -5 & \sqrt{6} & 0 & 0 \end{pmatrix}, \qquad [A, K_{2,4}] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & -5 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & \sqrt{6} \\ 0 & -5 & 0 & \sqrt{6} & 0 \end{pmatrix}.$$

Let G be the graph of A, and let M be the matrix defined in part (b) of Theorem 31 whose columns are $\mathrm{vec}_{\overline{G}}([A, K_{ij}])$. The submatrix $\widehat{M}$ of columns of M corresponding to K_{ij} where $(i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ is

$$\widehat{M} = \begin{pmatrix} -3 & -1 & 1 & 0 \\ -1 & -3 & 0 & 1 \\ 1 & 0 & -3 & -1 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

Since $\widehat{M}$ is strictly diagonally dominant (equivalently, 0 is not in the union of Gershgorin discs of $\widehat{M}$), $\widehat{M}$ is invertible, and so rank $\widehat{M} = 4$. Therefore M has rank 4 and by Theorem 31, we conclude that A has the SSP.

3.3 Gershgorin discs and the SSP

Given a square matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, the *Gershgorin intersection graph* of A is the graph on vertices labeled $1, \ldots, n$ in which two vertices $i \neq j$ are adjacent exactly when Gershgorin discs i and j of A intersect, that is, when the inequality

$$|a_{ii} - a_{jj}| \leqslant \sum_{\ell=1, \ell \neq i}^{n} |a_{i\ell}| + \sum_{\ell=1, \ell \neq j}^{n} |a_{j\ell}| \tag{13}$$

is satisfied. If A has real spectrum, then Gershgorin discs intersect if and only if they intersect on the real line, and the Gershgorin intersection graph of A is an interval graph.

Note that when graphs have a common vertex labeling, one of them may be a subgraph up to isomorphism of another without being identically a subgraph. The next result requires the stronger condition of being identically a subgraph.

Theorem 33. *Let G be a graph with vertices labeled $1, \ldots, n$ and let $A \in \mathcal{S}(G)$. If the Gershgorin intersection graph of A is identically a subgraph of G, then A satisfies the SSP.*

Proof. Suppose that $e_k = i_k j_k$ $(k = 1, 2, \ldots, p)$ are the edges of $\overline{G}$. Let $\widehat{M}$ be the $p \times p$ matrix whose k-th column is $\mathrm{vec}_{\overline{G}}(AK_{i_k, j_k} - K_{i_k, j_k}A)$.

The (k, k)-entry of $\widehat{M}$ has absolute value $|a_{i_k, i_k} - a_{j_k, j_k}|$ and the remaining entries of the k-th column of $\widehat{M}$ are, up to sign, a subset of the entries $a_{i_k, \ell}$, $\ell \neq i_k$, and $a_{j_k, \ell}$, $\ell \neq j_k$. If the Gershgorin intersection graph of A is identically a subgraph of G, then inequality (13) is not satisfied for any k (because the e_k are nonedges of G and therefore of the Gershgorin intersection graph). Thus $\widehat{M}$ is strictly diagonally dominant, so $\widehat{M}$ is invertible and has rank p. Therefore, by Theorem 31, A has the SSP. $\square$

Of course it is possible to have $\widehat{M}$ strictly diagonally dominant implying the invertibility of $\widehat{M}$ even when the Gershgorin intersection graph of A is not a subgraph of G, as in Example 32.

3.4 Block diagonal matrices

Theorem 34. *Let $A_i \in S_{n_i}(\mathbb{R})$ for $i = 1, 2$. Then $A := A_1 \oplus A_2$ has the SSP (respectively, SMP) if and only if both A_1 and A_2 have the SSP (respectively, SMP) and $\mathrm{spec}(A_1) \cap \mathrm{spec}(A_2) = \emptyset$.*

Proof. Let $X = \begin{pmatrix} X_1 & W \\ W^\top & X_2 \end{pmatrix}$ be partitioned conformally with A.

First, suppose that A_1 and A_2 have the SSP, $\mathrm{spec}(A_1) \cap \mathrm{spec}(A_2) = \emptyset$, $A \circ X = O, I \circ X = O$, and $[A, X] = O$. Since $A_i \circ X_i = O$, $I \circ X_i = O$, and $[A_i, X_i] = O$ for $i = 1, 2$ and A_i has SSP, $X_i = O$ for $i = 1, 2$. The 1,2-block of $[A, X]$ is $O = A_1 W - W A_2$, so by [20, Theorem 2.4.4.1], $W = O$. Thus $X = O$ and A has the SSP.

Now, suppose A_1 and A_2 have the SMP rather than the SSP (and the spectra are disjoint). Assume $A \circ X = O, I \circ X = O, [A, X] = O$ and X also satisfies $\mathrm{tr}(A^k X) = 0$ for $k = 2, \ldots, n - 1$. As before, $W = O$ so $X = X_1 \oplus X_2$, $A_i \circ X_i = O$, $I \circ X_i = O$, $[A_i, X_i] = O$ for $i = 1, 2$. To obtain $X_i = O, i = 1, 2$ (and thus $X = O$ and A has the SMP) it suffices to show that $\mathrm{tr}(A_i^k X_i) = 0$ for $i = 1, 2$ and $k = 2, \ldots, n - 1$. Consider the spectral decompositions of the diagonal blocks of A, $A_i = \sum_{j=1}^{q_i} \lambda_j^{(i)} E_j^{(i)}$ for $i = 1, 2$. Since $\mathrm{spec}(A_1) \cap \mathrm{spec}(A_2) = \emptyset$, the spectral decomposition of A is

$$A = \sum_{j=1}^{q_1} \lambda_j^{(1)} \left(E_j^{(1)} \oplus O \right) + \sum_{j=1}^{q_2} \lambda_j^{(2)} \left(O \oplus E_j^{(2)} \right).$$

Since each projection in the spectral decomposition is a polynomial in A, $A_1 \oplus O$ and $O \oplus A_2$ are polynomials in A. Therefore, $\text{tr}(A_1^k X_1) = \text{tr}\left(\left(A_1^k \oplus O\right) X\right) = 0$ and $\text{tr}(A_2^k X_2) = 0$.

Conversely, assume $A_1 \oplus A_2$ has the SSP and $A_1 \circ X_1 = O$, $I \circ X_1 = O$, and $[A_1, X_1] = O$. Then $X := \begin{pmatrix} X_1 & O \\ O & O \end{pmatrix}$ satisfies $A \circ X = O$, $I \circ X = O$, and $[A, X] = O$, so $X = O$, implying $X_1 = O$. In the case of the SMP, $\text{tr}(A_1^k X_1) = 0$ implies $\text{tr}(A^k X) = 0$. We show that $\text{spec}(A_1) \cap \text{spec}(A_2) \neq \emptyset$ implies A does not have the SMP (and thus does not have the SSP): Suppose $\lambda \in \text{spec}(A_1) \cap \text{spec}(A_2)$. For $i = 1, 2$, choose $\mathbf{z}_i \neq 0$ such that $A_i \mathbf{z}_i = \lambda \mathbf{z}_i$. Define

$$Z := \begin{pmatrix} O & \mathbf{z}_1 \mathbf{z}_2^\top \\ \mathbf{z}_2 \mathbf{z}_1^\top & O \end{pmatrix}.$$

Then $A \circ Z = O$, $I \circ Z = O$, $\text{tr}(A^k Z) = 0$, and $[A, Z] = O$, but $Z \neq O$, showing that A does not have the SMP. $\square$

As one application we give an upper bound on $q(G)$ in terms of chromatic numbers.

Theorem 35. *Let G be a graph and $\overline{G}$ its complement. Then $q(G) \leqslant 2\chi(\overline{G})$.*

Proof. The graph G contains a disjoint union of $\chi(\overline{G})$ cliques. Taking the direct sum of realizations of each clique each having at most two distinct eigenvalues and the SSP, and the eigenvalues of different cliques distinct gives a matrix having the SSP by Theorem 34. The result then follows from Theorem 10. $\square$

Another application gives an upper bound on the number of distinct eigenvalues required for a supergraph on a superset of vertices.

Theorem 36. *Let A be a symmetric matrix of order n with graph G. If A has the SSP (or the SMP) and $\widehat{G}$ is a graph on m vertices containing G as a subgraph, then there exists $\widehat{B} \in \mathcal{S}(\widehat{G})$ such that $\text{spec}(A) \subseteq \text{spec}(\widehat{B})$ (or has the ordered multiplicity list of A augmented with ones), and $\widehat{B}$ has the SSP (or the SMP). Furthermore, $q(\widehat{G}) \leqslant m - n + q(A)$. If A has the SSP, then we can prescribe $\text{spec}(\widehat{B})$ to be $\text{spec}(A) \cup \Lambda$ where Λ is any set of distinct real numbers such that $\text{spec}(A) \cap \Lambda = \emptyset$.*

Proof. Assume that A has the SSP (respectively, SMP), and without loss of generality that $V(G) = \{1, 2, \ldots, n\}$, $V(\widehat{G}) = \{1, 2, \ldots, m\}$, and G is a subgraph of $\widehat{G}$.

Consider the matrix

$$B = \begin{pmatrix} A & O \\ O & \text{diag}(\Lambda) \end{pmatrix}.$$

Note that the eigenvalues of $\text{diag}(\Lambda)$ are distinct and distinct from the eigenvalues of A. It follows that $\text{diag}(\Lambda)$ has the SSP (see Remark 15 or note this follows from Theorem 34), and thus has the SMP. By Theorem 34, if A satisfies the SSP (SMP), then B satisfies the SSP (SMP).

By Theorem 10 (or Theorem 20), every supergraph of the graph of B on the same vertex set has a realization $\widehat{B}$ that is cospectral with B and has the SSP (or has the same ordered multiplicity list and has the SMP). Hence $q(\widehat{G}) \leqslant q(B) = m - n + q(A)$. $\square$

Remark 37. By taking a realization in $\mathcal{S}(G)$ with row sums 0, $\mathrm{mr}(G) \leqslant |G| - 1$. It is a well known result that the eigenvalues of an irreducible tridiagonal matrix are distinct, that is $\mathrm{mr}(P_n) = n - 1$. A classic result of Fiedler [15] asserts that $\mathrm{mr}(G) = |G| - 1$ if and only if G is a path.

Theorem 36 can be used to derive this characterization, as follows. If G contains a vertex of degree 3 or more, then G contains $K_{1,3}$ as a subgraph, and hence by Theorem 36 and Example 11 we conclude that $\mathrm{mr}(G) \leqslant |G| - 2$. Also, it is easy to see if G is disconnected, then $\mathrm{mr}(G) \leqslant |G| - 2$. Thus, if $\mathrm{mr}(G) = |G| - 1$, then G has maximum degree 2 and is connected. Hence G is a path or a cycle. The adjacency matrix of a cycle C has a multiple eigenvalue, which implies that $\mathrm{mr}(C) \leqslant |C| - 2$.

4 Application of strong properties to minimum number of distinct eigenvalues

The SSP and the SMP allow us to characterize graphs having $q(G) \geqslant |G| - 1$ (see Section 4.2). First we introduce new parameters based on the minimum number of eigenvalues for matrices with the given graph that have the given strong property.

4.1 New parameters $q_S(G)$ and $q_M(G)$

Recall that $\xi(G)$ is defined as the maximum nullity among matrices in $\mathcal{S}(G)$ that satisfy the SAP, and $\nu(G)$ is the maximum nullity of positive semidefinite matrices having the SAP, so $\xi(G) \leqslant \mathrm{M}(G)$ and $\nu(G) \leqslant \mathrm{M}_+(G)$. These bounds are very useful because of the minor monotonicity of ξ and ν (especially the monotonicity on subgraphs). In analogy with these definitions, we define parameters for the minimum number of eigenvalues among matrices having the SSP or the SMP and described by a given graph. In order to do this we need the property that every graph has at least one matrix with the property SSP (and hence SMP). For any set of $|G|$ distinct real numbers, there is matrix in $\mathcal{S}(G)$ with these eigenvalues that has the SSP by Remark 15.

Definition 38. We define

$$q_M(G) = \min\{q(A) : A \in \mathcal{S}(G) \text{ and } A \text{ has the SMP}\}$$

and

$$q_S(G) = \min\{q(A) : A \in \mathcal{S}(G) \text{ and } A \text{ has the SSP}\}.$$

Observation 39. *From the definitions, $q(G) \leqslant q_M(G) \leqslant q_S(G)$ for any graph G.*

One might ask why we have not defined a parameter $q_A(G)$ for the SAP. The reason is that the SAP is not naturally associated with the minimum number of eigenvalues. The Strong Arnold Property considers only the eigenvalue zero; that is, if zero is a simple eigenvalue of A (or not an eigenvalue of A), then A automatically has the SAP. The next result is immediate from Remark 19 and the fact that $q(G) = 1$ if and only if G has no edges.

Corollary 40. *Suppose G is connected. Then $q_S(G) = 2$ if and only if $q_M(G) = 2$.*

The next result is immediate from Theorem 34.

Corollary 41. *If G is the disjoint union of connected components $G_i, i = 1, \ldots, h$ with $h \geqslant 2$, then $q_S(G) = \sum_{i=1}^{h} q_S(G_i)$ and $q_M(G) = \sum_{i=1}^{h} q_M(G_i)$.*
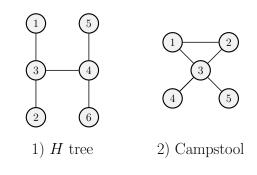
Remark 42. Suppose G is the disjoint union of connected components $G_i, i = 1, \ldots, h$ with $h \geqslant 2$. Since any graph has a realization for any set of distinct eigenvalues (Remark 15), $q(G) \leqslant \max_{i=1}^{h} |G_i|$. Clearly $q(G) \geqslant \max_{i=1}^{h} q(G_i)$.

The next result is an immediate corollary of Theorem 36.

Corollary 43. *If H is a subgraph of G, $|H| = n$, and $|G| = m$, then $q(G) \leqslant q_S(G) \leqslant m - n + q_S(H)$ and $q(G) \leqslant q_M(G) \leqslant m - n + q_M(H)$.*

4.2 High values of $q(G)$

In this section we characterize graphs having $q(G) \geqslant |G| - 1$.

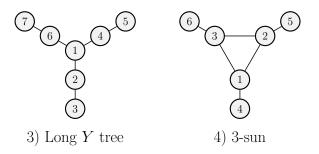1) H tree   2) Campstool

3) Long Y tree   4) 3-sun

Figure 1: Graphs for Proposition 44.

Proposition 44. *Let G be one of the graphs shown in Figure 1. Then $q_S(G) \leqslant |G| - 2$.*

Proof. Let

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \text{ and } A_4 = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The graphs of matrices A_1, A_2, A_3 and A_4 are the H tree, the campstool, the long Y tree, and the 3-sun, respectively. Also, $q(A_i) = |G| - 2$ for $i = 1, 2, 3, 4$. It is straightforward to verify that each of the matrices A_1, A_2, A_3 and A_4 has the SSP (see, for example, [26]). □

Corollary 45. *If a graph G contains a subgraph isomorphic to the H tree, the campstool, the long Y tree, or the 3-sun then*

$$q(G) \leqslant q_S(G) \leqslant |G| - 2.$$

Proof. This follows from Corollary 43 and Proposition 44. □

The parameters $\mathrm{M}(G)$ and $\mathrm{M}_+(G)$ can be used to construct lower bounds on $q(G)$. For a graph G of order n, clearly $q(G) \geqslant \left\lceil \frac{n}{\mathrm{M}(G)} \right\rceil$. The next result improves on this in many cases, in particular for $G = K_{1,3}$.

Proposition 46. *For any graph G on n vertices,*

$$q(G) \geqslant 2 + \left\lceil \frac{n - 2\mathrm{M}_+(G)}{\mathrm{M}(G)} \right\rceil.$$

Moreover, if $\mathrm{M}_+(G) < \frac{n}{2}$, then $q(G) \geqslant 3$.

Proof. Let $A \in \mathcal{S}(G)$ be a matrix with $q(G)$ distinct eigenvalues. Let α, β be the smallest and the largest eigenvalues of A. Since $A - \alpha I$ and $-A + \beta I$ are positive semidefinite, the multiplicity of α and β is no more than $\mathrm{M}_+(G)$. Every other eigenvalue of A has multiplicity less than or equal to $\mathrm{M}(G)$. Therefore, A has at least $2 + \left\lceil \frac{n - 2\mathrm{M}_+(G)}{\mathrm{M}(G)} \right\rceil$ distinct eigenvalues. The final statement of the proposition readily follows. □

Corollary 47. *If G contains two vertex disjoint subgraphs each of which is a K_3 or a $K_{1,3}$, then $q(G) \leqslant |G| - 2$.*

Proof. The 3×3 all ones matrix J_3 has two distinct eigenvalues and has SSP, so $q_S(K_3) = 2$. Example 11 and Proposition 46 imply that $q_S(K_{1,3}) = 3$. Thus, by Corollary 41 and Corollary 43, $q(G) \leqslant |G| - 2$. □

In Ferguson [16, Theorem 4.3] it is shown that a multiset of n real numbers is the spectrum of an $n \times n$ periodic Jacobi matrix[5] A if and only if these numbers can be arranged as $\lambda_1 > \lambda_2 \geqslant \lambda_3 > \lambda_4 \geqslant \lambda_5 > \cdots \lambda_n$. This solves the inverse eigenvalue problem for a cycle of odd length: Suppose $A \in \mathcal{S}(C_n)$ and n is odd. If the cycle product $a_{12}a_{23}\cdots a_{n-1,n}a_{n1} > 0$, then A is similar (by a diagonal matrix with diagonal entries in $\{\pm 1\}$) to a periodic Jacobi matrix; if $a_{12}a_{23}\cdots a_{n-1,n}a_{n1} < 0$, then $-A$ is similar to a periodic Jacobi matrix.

In the next result we establish that a specific matrix $A \in \mathcal{S}(C_n)$ with $q(A) = \lceil \frac{n}{2} \rceil$ has the SMP.

Theorem 48. *Let C_n be the cycle on $n \geqslant 3$ vertices. Then $q_M(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Since $\mathrm{M}(C_n) = 2$, $q(C_n) \geqslant \lceil \frac{n}{2} \rceil$. Given $n \geqslant 3$, let $C = (c_{ij})$ be the *flipped-cycle matrix* of order n, that is, the (non-symmetric) $n \times n$ matrix with entries $c_{i,i+1} = 1$ for $i = 1, \ldots, n - 1$, $c_{n,1} = -1$, and 0 otherwise. Since C satisfies the equation $C^n = -I$, the eigenvalues of C are the nth roots of -1. The matrix $A = C + C^\top = C + C^{-1}$ is a symmetric matrix whose graph is the n-cycle C_n, and whose eigenvalues are $2\cos(2\pi \frac{2j-1}{2n})$ for $j \in \{1, \ldots, n\}$, which occur in $\lfloor \frac{n}{2} \rfloor$ pairs satisfying $j_1 + j_2 = n + 1$, with one singleton eigenvalue (coming from $2j = n + 1$) equal to -2 when n is odd. Thus $q(A) = \lceil \frac{n}{2} \rceil$.

We show that A has the SMP, implying $q_M(C_n) = \lceil \frac{n}{2} \rceil$. Assume $X = (x_{ij})$ is a symmetric matrix such that $A \circ X = O$, $I \circ X = O$, $[A, X] = O$, and $\mathrm{tr}(A^k X) = 0$ for $k = 1, \ldots, n$. Divide the entries x_{ij} (on or above the main diagonal) into n bands for $k = 0, \ldots, n - 1$ of the form $x_{i,i+k}$, $i = 1, \ldots, n - k$; all the entries of X in bands 0, 1 and $n - 1$ are zero since $I \circ X = O$ and $A \circ X = O$. The fact that AX is symmetric implies that all the entries in each band are equal, and in addition, $x_{1,1+(n-k)} = -x_{1,1+k}$ for $k \leqslant \frac{n}{2}$. In the case that $n = 2\ell$ is even, this implies $x_{i,i+\ell} = 0$. Now assume $X \neq O$ and let m be the smallest natural number such that band m of X contains a nonzero entry x_{ij} (and $2 \leqslant m < \frac{n}{2}$). Notice that the sign pattern $x_{i,i+m} = x_{1,1+m}$ for $i = 1, \ldots, n - m$ and $x_{i,i+(n-m)} = -x_{1,1+m}$ for $i = 1, \ldots, m$ matches the sign pattern of A^m, i.e., $(A^m)_{i,i+m} = 1$ for $i = 1, \ldots, n - m$ and $(A^m)_{i,i+(n-m)} = -1$ for $i = 1, \ldots, m$. It is clear that $(A^m)_{ij} = 0$ when the distance between i and j is greater than m, and by the choice of m, $x_{ij} = 0$ when the distance between i and j is less than m. Thus $\mathrm{tr}(A^m X) = 2nx_{1,1+m} \neq 0$, a contradiction. □

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[5]A periodic Jacobi matrix is a real symmetric matrix with nonnegative off-diagonal entries whose graph is a cycle.

By the proof of Theorem 48 and Remark 19, the symmetric flipped cycle matrix $C+C^\top$ has the SSP for $n = 4$. It is clear from $X = C^2 + C^{2\top}$ that the symmetric flipped cycle matrix $C + C^\top$ does not have the SSP for $n \geqslant 5$ (this is implicit in the the proof of Theorem 48).

The next corollary is immediate from Corollary 43 and Theorem 48.

Corollary 49. *If a graph G contains a cycle of length k, then*

$$q(G) \leqslant q_M(G) \leqslant |G| - \left\lfloor \frac{k}{2} \right\rfloor.$$

Proposition 50. *Let G be a graph. Then the following are equivalent:*

(a) $q(G) = |G|$,

(b) $\mathrm{M}(G) = 1$,

(c) G *is a path.*

Proof. The equivalence of (b) and (c) is shown in [15] (see also Remark 37). To show the equivalence of (a) and (b), first assume that $q(G) = |G|$. By Proposition 2.5 in [1], $q(G) \leqslant \mathrm{mr}(G) + 1$, so $\mathrm{mr}(G) = |G| - 1$ and (b) holds. Conversely, if $\mathrm{M}(G) = 1$, then every eigenvalue has multiplicity at most one, so $q(G) = |G|$. $\square$

The next result characterizes all of the graphs G that satisfy $q(G) \geqslant |G| - 1$ and resolves a query presented in [1] on connected graphs that satisfy $q(G) = |G| - 1$.

Theorem 51. *A graph G has $q(G) \geqslant |G| - 1$ if and only if G is one of the following:*

(a) *a path,*

(b) *the disjoint union of a path and an isolated vertex,*

(c) *a path with one leaf attached to an interior vertex,*

(d) *a path with an extra edge joining two vertices at distance 2.*

Proof. Suppose G is a graph with $q(G) \geqslant |G| - 1$. By [1, Proposition 2.5], $q(G) \leqslant \mathrm{mr}(G) + 1$, so $\mathrm{mr}(G) \geqslant |G| - 2$ and $\mathrm{M}(G) \leqslant 2$.

If G has connected components $G_i, i = 1, \dots h$, with $h \geqslant 3$ or at least two components containing two or more vertices, then by Remark 42, $q(G) \leqslant \max |G_i| \leqslant |G| - 2$. If $G = H \dot\cup K_1$, then $|H| = |G| - 1 = q(G) = q(H)$, which implies H is a path by Proposition 50. So henceforth we assume G is connected.

If G has at least 2 cycles, then G contains as a subgraph at least one of C_n with $n \geqslant 4$, two disjoint copies of K_3, or the campstool. By Corollaries 49, 47, 45, it is impossible to have any of these graphs as a subgraph. Thus G has one C_3 and no other cycles, or G is a tree.

Suppose that G has one 3-cycle. Since the 3-sun is forbidden as a subgraph of G by Corollary 45, at least one vertex of the C_3 has degree two; let v be this vertex. If $G - v$ is not a path, then $G - v$ has a vertex w of degree 3 or more. Assume first that w is not on the C_3 and note that w is a cut-vertex of G. By choosing singular matrices for each component of $G - w$ and the matrix for the component containing v having nullity two, we see that $\mathrm{M}(G - w) \geqslant 4$, leading to the contradiction that $\mathrm{M}(G) \geqslant 3$. Now suppose w is a vertex of C_3. Then G contains the campstool, contradicting Corollary 45. Hence $G - v$ is a path, and G consists of a path together with an extra edge joining two vertices at distance 2.

Now suppose that G is a tree. Since $\mathrm{M}(G) \leqslant 2$, the maximum degree of G is at most three by a theorem of Parter and Wiener (see, for example, [13, Theorem 2.1]). If the maximum degree is 2, then G is a path. Otherwise, let w be a vertex of degree three. Then $\mathrm{M}(G) \leqslant 2$ implies that each component of $G - w$ must be a path. Since the H tree is a forbidden subgraph by Corollary 45, the vertex of each path adjacent to w is an endpoint of the path. Since the long Y tree is also forbidden by Corollary 45, at least one of the components of $G - w$ is an isolated vertex, and we conclude that G is path along with a pendent edge attached to an interior vertex of the path.

Each of the graphs G listed in the statement of the theorem has a unique path between two vertices of distance $|G| - 2$, so by [1, Theorem 3.2], $q(G) \geqslant |G| - 1$. $\qquad\square$

5 Conclusions

We have presented a careful development of the theory associated with the SSP and SMP as useful extensions of the SAP. A major consequence is that for any given graph G of order n and multiset of n real numbers (or partition of n), if one can find an SSP (or SMP) matrix $A \in \mathcal{S}(G)$ with spectrum equal to this multiset (or multiplicity list equal to this partition), then for each supergraph $\widetilde{G}$ on the same vertex set as G, that same spectrum (or multiplicity list) is attained by some matrix in $\mathcal{S}(\widetilde{G})$. Further implications to the inverse eigenvalue problem for graphs are expected, e.g., using these tools it may be possible to solve the inverse eigenvalue problem for additional families of graphs. A key issue moving forward, partly addressed here, is detection and construction of matrices in $\mathcal{S}(G)$ that have the SSP or SMP.

One of our main motivations for considering the SSP and the SMP was for evaluating $q(G)$. In this work, we concentrated on the case of graphs G for which $q(G)$ was large relative to $|G|$. We established a number of forbidden subgraph-type results that were used to characterize the graphs G such that $q(G) \geqslant |G| - 1$. As a consequence, we produced a new verification of Fiedler's characterization that the path is the only graph G such that $q(G) = |G|$, and we resolved an open problem left from the work in [1] concerning graphs G with $q(G) = |G| - 1$. A much clearer picture of $q(G)$ may be obtained by using the tools and techniques derived here.

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