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Using a new zero forcing process to guarantee the Strong Arnold Property

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ABSTRACT

The maximum nullity $M(G)$ and the Colin de Verdière type parameter $\xi(G)$ both consider the largest possible nullity over matrices in $\mathcal{S}(G)$, which is the family of real symmetric matrices whose i,j-entry, $i \neq j$, is nonzero if i is adjacent to j, and zero otherwise; however, $\xi(G)$ restricts to those matrices A in $\mathcal{S}(G)$ with the Strong Arnold Property, which means $X = O$ is the only symmetric matrix that satisfies $A \circ X = O$, $I \circ X = O$, and $AX = O$. This paper introduces zero forcing parameters $Z_{\mathrm{SAP}}(G)$ and $Z_{\mathrm{vc}}(G)$, and proves that $Z_{\mathrm{SAP}}(G) = 0$ implies every matrix $A \in \mathcal{S}(G)$ has the Strong Arnold Property and that the inequality $M(G) - Z_{\mathrm{vc}}(G) \leq \xi(G)$ holds for every graph G. Finally, the values of $\xi(G)$ are computed for all graphs up to 7 vertices, establishing $\xi(G) = \lfloor Z \rfloor(G)$ for these graphs.

1. Introduction

A *minimum rank problem* for a graph G is to determine what is the smallest possible rank, or equivalently the largest possible nullity, among a family of matrices associated with G. One classical way to associate matrices to a graph G is through $\mathcal{S}(G)$, which is

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defined as the set of all real symmetric matrices whose i, j-entry, $i \neq j$, is nonzero whenever i and j are adjacent in G, and zero otherwise. Note that the diagonal entries can be any real number. Another association is $\mathcal{S}_+(G)$, which is the set of positive semidefinite matrices in $\mathcal{S}(G)$. Thus, the *maximum nullity* $M(G)$ and the *positive semidefinite maximum nullity* $M_+(G)$ are defined as

$$M(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\}, \text{ and}$$
$$M_+(G) = \max\{\text{null}(A) : A \in \mathcal{S}_+(G)\}.$$

The classical minimum rank problem is a branch of the *inverse eigenvalue problem*, which asks for a given multi-set of real numbers, is there a matrix in $\mathcal{S}(G)$ such that its spectrum is composed of these real numbers. If λ is an eigenvalue of some matrix $A \in \mathcal{S}(G)$, then its multiplicity should be no higher than $M(G)$, for otherwise $A - \lambda I$ has nullity higher than $M(G)$. Similarly, $M_+(G)$ provides an upper bound for the multiplicities of the smallest and the largest eigenvalues. Also, $M_+(G)$ is closely related to faithful *orthogonal representations* [12].

Other families of matrices are defined through the Strong Arnold Property. A matrix A is said to have the *Strong Arnold Property* (or SAP) if the zero matrix is the only symmetric matrix X that satisfies the three conditions $A \circ X = O$, $I \circ X = O$, and $AX = O$. Here I and O are the identity matrix and the zero matrix of the same size as A, respectively, and $\circ$ is the Hadamard (entrywise) product of matrices. By adding the SAP to the conditions of the abovementioned families, the *Colin de Verdière type parameters* are defined as

$$\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has the SAP}\} \text{ [5], and}$$
$$\nu(G) = \max\{\text{null}(A) : A \in \mathcal{S}_+(G), A \text{ has the SAP}\} \text{ [8].}$$

These parameters are variations of the original Colin de Verdière parameter $\mu(G)$ [7], which is defined as the maximum nullity over matrices A such that

- $A \in \mathcal{S}(G)$ and every off-diagonal entry of A is non-positive (called a *generalized Laplacian of* G),
- A has exactly one negative eigenvalue including the multiplicity, and
- A has the SAP.

In order to see how the SAP makes a difference between these parameters, we define $M_\mu(G)$ as the maximum nullity of the same family of matrices by ignoring the SAP, i.e. the maximum nullity of a generalized Laplacian A of G such that A has exactly one negative eigenvalue.

The SAP gives $\xi(G)$, $\nu(G)$, and $\mu(G)$ nice properties. For example, they are *minor monotone* [12]. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of deleting edges, deleting vertices, and contracting edges; a graph parameter ζ

is said to be minor monotone if $\zeta(H) \leq \zeta(G)$ whenever H is a minor of G. By the graph minor theorem (e.g., see [10]), for a given integer d and a minor monotone parameter ζ, the minimal forbidden minors for $\zeta(G) \leq d$ consist of only finitely many graphs. Here ζ can be ξ, ν or μ. More specifically, $\mu(G) \leq 3$ if and only if G is a planar graph [18], which is characterized by the forbidden minors K_5 and $K_{3,3}$.

However, the SAP also makes the Colin de Verdière type parameters less controllable by the existing tools. For example, zero forcing parameters, which will be defined in Section 1.2, were used extensively as a bound for the minimum rank problem. For the classical zero forcing number $Z(G)$, it is known that $M(G) \leq Z(G)$ for all graphs [2]; and $M(G) = Z(G)$ when G is a tree or $|V(G)| \leq 7$ [2,9]. An analogy for $\xi(G)$ is the minor monotone floor of the zero forcing number, which is denoted as $\lfloor Z \rfloor(G)$ and will be defined in Section 4. It is known that $\xi(G) \leq \lfloor Z \rfloor(G)$ for all graphs [4]. The similar statement $\xi(G) = \lfloor Z \rfloor(G)$ is not always true when G is a tree [4], and no results about $\xi(G)$ and $\lfloor Z \rfloor(G)$ for small graphs are known.

The main goal of this paper is to establish a connection between zero forcing parameters and the SAP, and derive consequences. This leads to some questions. Does some graph structure guarantee that every $A \in \mathcal{S}(G)$ has the SAP? Thus, the maximum nullity does not change when the SAP condition is added. Specifically, when A is a generalized Laplacian of some graph G and has exactly one negative eigenvalue, some graph structures do guarantee that A has the SAP [15,21]; however, general results on this problem remain unknown. On the other hand, is there a strategy to perturb any given matrix such that it guarantees the SAP? Thus, the rank changed by the perturbation gives an upper bound for $M(G) - \xi(G)$.

In Section 2, we introduce a new parameter $Z_{\mathrm{SAP}}(G)$ and its variants Z^{ℓ}_{SAP} and Z^{+}_{SAP}, and prove in Theorem 2.6 that under the condition $Z_{\mathrm{SAP}}(G) = 0$, every matrix $A \in \mathcal{S}(G)$ has the SAP. Thus, $\xi(G) = M(G)$, $\nu(G) = M_{+}(G)$, and $\mu(G) = M_{\mu}(G)$ when $Z_{\mathrm{SAP}}(G) = 0$, so finding the values of Colin de Verdière type parameters is equivalent to finding the values of the corresponding parameters. Table 1 in Section 2.2 indicates that there are actually a considerable proportion of graphs that have this property.

In Section 3, another parameter $Z_{\mathrm{vc}}(G)$ and its variant $Z^{\ell}_{\mathrm{vc}}(G)$ are defined, and Theorem 3.2 states that $M(G) - \xi(G) \leq Z_{\mathrm{vc}}(G)$ for every graph G. With the help of $Z_{\mathrm{SAP}}(G)$, $Z_{\mathrm{vc}}(G)$, and some existing theorems, Section 4 provides the result that $\xi(G) = \lfloor Z \rfloor(G)$ for graphs G up to 7 vertices.

All parameters introduced in this paper and their relations are illustrated in Fig. 1. A brief description of the related theorems is given on the sides. A line between two parameters means the lower one is less than or equal to the upper one.

Throughout the paper, the neighborhood of a vertex i in a graph G is denoted as $N_G(i)$, while the closed neighborhood is denoted as $N_G[i]$, which equals $N_G(i) \cup \{i\}$. The induced subgraph on a vertex set W of G is denoted as $G[W]$. If A is a matrix, U and W are subsets of the row and column indices of A respectively, then $A[U, W]$ is the submatrix of A induced on the rows of U and columns of W; if U and W are ordered sets, then permute the rows and columns of this submatrix accordingly.

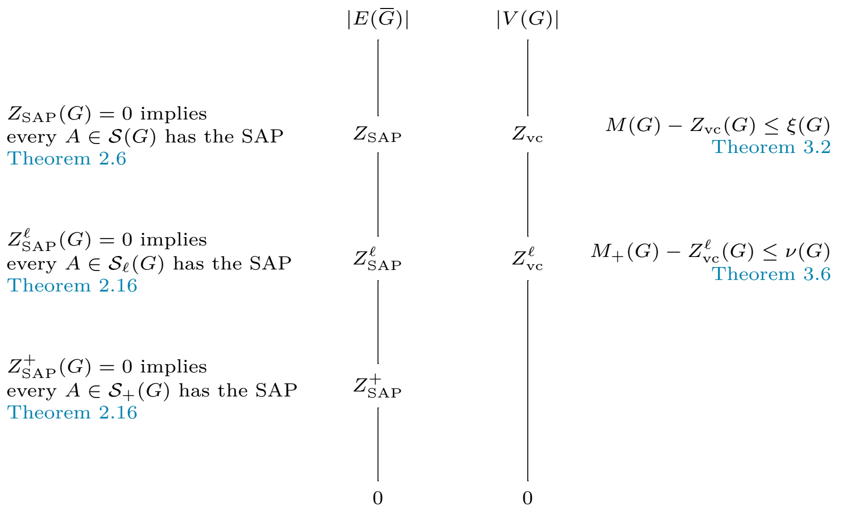


Fig. 1. Parameters introduced in this paper.

1.1. SAP system and its matrix representation

Let G be a graph on n vertices, and $\overline{m} = |E(\overline{G})|$. In order to see if a matrix $A \in \mathcal{S}(G)$ has the SAP or not, the matrix X can be viewed as a symmetric matrix with $\overline{m}$ variables at the positions of non-edges so that X satisfies $A \circ X = I \circ X = O$. Next, $AX = O$ leads to n^2 restrictions on the $\overline{m}$ variables, which forms a linear system. Call this linear system the *SAP system of* A, which can also be written as an $n^2 \times \overline{m}$ matrix.

Definition 1.1. Let G be a graph on n vertices, $\overline{m} = |E(\overline{G})|$, and $A = [\,a_{i,j}\,] \in \mathcal{S}(G)$. Given an order of the set of non-edges, the *SAP matrix of* A with respect to this order is an $n^2 \times \overline{m}$ matrix Ψ whose rows are indexed by pairs (i,k) and columns are indexed by the non-edges $\{j,h\}$ such that

$$\Psi_{(i,k),\{j,h\}} = \begin{cases} 0 & \text{if } k \notin \{j,h\}, \\ a_{i,j} & \text{if } k \in \{j,h\} \text{ and } k = h. \end{cases}$$

The rows follow the order $(i,k) < (j,h)$ if and only if $k < h$, or $k = h$ and $i < j$; the columns follow the order of the non-edges.

Remark 1.2. Let G be a graph, $A \in \mathcal{S}(G)$, and Ψ the SAP matrix of A with respect a given order of the non-edges. The columns of Ψ correspond to the $\overline{m}$ variables in X, and the row for (i,j) represents the equation $(AX)_{i,j} = 0$. Therefore, a matrix has the SAP if and only if the corresponding SAP matrix is full-rank.

The rows of Ψ can be partitioned into n blocks, each having n elements. The k-th block are those rows indexed by (i,k) for $1 \leq i \leq n$. Let $\mathbf{v}_j$ be the j-th column of A. For the submatrix of Ψ induced by the rows in the k-th block, the $\{j,h\}$ column is $\mathbf{v}_j$ if

$k \in \{j,h\}$ and $k = h$, and is a zero vector otherwise. Equivalently, on the $\{i,j\}$ column of Ψ, the i-th block is $\mathbf{v}_j$, the j-th block is $\mathbf{v}_i$, while other blocks are zero vectors.

Example 1.3. Let $G = P_4$ be the path on four vertices, with the vertices labeled by $\{1,2,3,4\}$ in the path order. Consider a matrix $A \in \mathcal{S}(G)$ and the matrix X with three variables, as shown below.

$$AX = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & x_{\{1,3\}} & x_{\{1,4\}} \\ 0 & 0 & 0 & x_{\{2,4\}} \\ x_{\{1,3\}} & 0 & 0 & 0 \\ x_{\{1,4\}} & x_{\{2,4\}} & 0 & 0 \end{bmatrix} \end{array}$$

The SAP matrix of A with respect to the order $(\{1,3\},\{1,4\},\{2,4\})$ is a matrix Ψ representing the linear system for $AX = O$ with three variables $x_{\{1,3\}}, x_{\{1,4\}}, x_{\{2,4\}}$. For convenience, write $A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4\,]$, where $\mathbf{v}_j$ is the j-th column vector of A. Now $AX = O$ means

$$\sum_{j \notin N_G[k]} x_{\{j,k\}} \mathbf{v}_j = \mathbf{0} \text{ for each } k \in V(G).$$

Thus,

$$\Psi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array} = \begin{array}{c} \\ (1,1) \\ (2,1) \\ (3,1) \\ (4,1) \\ (1,2) \\ (2,2) \\ (3,2) \\ (4,2) \\ (1,3) \\ (2,3) \\ (3,3) \\ (4,3) \\ (1,4) \\ (2,4) \\ (3,4) \\ (4,4) \end{array} \begin{array}{c} \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

1.2. Zero forcing parameters

On a graph G, the conventional *zero forcing game* (ZFG) is a color-change game such that each vertex is colored blue or white initially, and then the *color change rule* (CCR) is applied repeatedly. If starting with an initial blue set $B \subseteq V(G)$ and every vertex turns blue eventually, this set B is called a *zero forcing set* (ZFS). The zero forcing number is defined as the minimum cardinality of a ZFS.

Different types of zero forcing numbers are discussed in the literature (e.g., see [3,4,12]). Most of them serve as upper bounds of different types of maximum nullities. Here we consider three types of the zero forcing numbers Z, Z_ℓ, Z_+ with the corresponding color change rules:

- (CCR-Z) If i is a blue vertex and j is the only white neighbor of i, then j may turn blue.
- (CCR-Z_ℓ) CCR-Z can be used to perform a force. Or if i is a white vertex without white neighbors and i is not isolated, then i may turn blue.
- (CCR-Z_+) Let B be the set of blue vertices at some stage and W the vertices of a component of $G - B$. CCR-Z is applied to $G[B \cup W]$ with blue vertices B.

When a zero forcing game is mentioned, it is equipped with a color change rule, and we use $i \to j$ to denote a corresponding force (i.e. i forcing j to become blue). Note that for CCR-Z_ℓ, it is possible to have $i \to i$. Also, at the same stage, the color change rule might be able to apply to different i and j (or W for CCR-Z_+), so the player has the choice to decide where to apply the rule, though the final coloring where no more color change rules can be applied is always the same.

It is known [2–4] that

$$M(G) \le Z(G), \qquad M_+(G) \le Z_+(G), \quad \text{and} \quad Z_+(G) \le Z_\ell(G) \le Z(G).$$

Denote $\mathcal{S}_\ell(G)$ as those matrices in $\mathcal{S}(G)$ whose i,i-entry is zero if and only if vertex i is an isolated vertex. Then every matrix $A \in \mathcal{S}_\ell(G)$ has nullity at most $Z_\ell(G)$ [13].

All these results rely on Proposition 1.4.

Proposition 1.4. [2,3,13] *Let G be a graph on n vertices. Suppose at some stage B is the set of blue vertices.*

- *If $i \to j$ under CCR-Z, then for any matrix $A \in \mathcal{S}(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s \notin B} x_s \mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*
- *If $i \to j$ under CCR-Z_ℓ, then for any matrix $A \in \mathcal{S}_\ell(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s \notin B} x_s \mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*
- *If $i \to j$ under CCR-Z_+, then for any matrix $A \in \mathcal{S}_+(G)$ with column vectors $\{\mathbf{v}_s\}_{s=1}^n$, $\sum_{s \notin B} x_s \mathbf{v}_s = \mathbf{0}$ implies $x_j = 0$.*

2. SAP zero forcing parameters

In this section, we introduce a new parameter $Z_{\mathrm{SAP}}(G)$ and prove that if $Z_{\mathrm{SAP}}(G) = 0$ then every matrix $A \in \mathcal{S}(G)$ has the SAP, which implies $M(G) = \xi(G)$. We also introduce similar parameters and results for other variants.

First we give two examples illustrating what we called in Definition 2.4 the forcing triple and the odd cycle rule.

Example 2.1. Consider the graph P_4. Let A be the matrix as in Example 1.3 and $\mathbf{v}_j$ its j-th column. In Example 1.3, we know the SAP matrix of A can be written as

$$\begin{array}{cc} & \begin{array}{ccc} x_{\{1,3\}} & x_{\{1,4\}} & x_{\{2,4\}} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \end{array}.$$

Since $\mathbf{v}_4$ is the only nonzero vector on the second block-row, $x_{\{2,4\}}$ must be 0 in this linear system. Similarly, $\mathbf{v}_1$ is the only nonzero vector on the third block-row, so $x_{\{1,3\}} = 0$. Provided that $x_{\{1,3\}} = x_{\{2,4\}} = 0$, the structure on the first block-row forces $x_{\{1,4\}} = 0$. Since this argument holds for every matrix in $\mathcal{S}(G)$, every matrix in $\mathcal{S}(G)$ has the SAP.

Example 2.2. Let $G = K_{1,3}$. Consider the matrices A and X as

$$A = \begin{bmatrix} d_1 & a_1 & a_2 & a_3 \\ a_1 & d_2 & 0 & 0 \\ a_2 & 0 & d_3 & 0 \\ a_3 & 0 & 0 & d_4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_{\{2,3\}} & x_{\{2,4\}} \\ 0 & x_{\{2,3\}} & 0 & x_{\{3,4\}} \\ 0 & x_{\{2,4\}} & x_{\{3,4\}} & 0 \end{bmatrix}.$$

Let $\mathbf{v}_j$ be the j-th column of A. Then the SAP matrix of A with respect to the order $(\{2,3\},\{3,4\},\{2,3\})$ can be written as

$$\Psi = \begin{array}{cc} & \begin{array}{ccc} x_{\{2,3\}} & x_{\{3,4\}} & x_{\{2,4\}} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{v}_3 & \mathbf{0} & \mathbf{v}_4 \\ \mathbf{v}_2 & \mathbf{v}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_3 & \mathbf{v}_2 \end{bmatrix} \end{array}.$$

Recall that the row with index (i,j) is the i-th row in the j-th block. Thus the submatrix induced by rows $\{(1,2),(1,3),(1,4)\}$ is

$$\begin{bmatrix} a_2 & 0 & a_3 \\ a_1 & a_3 & 0 \\ 0 & a_2 & a_1 \end{bmatrix},$$

whose determinant is always nonzero if $a_1, a_2, a_3 \neq 0$. This means the SAP matrix of A is always full-rank, regardless the choice of $A \in \mathcal{S}(G)$. Hence every matrix $A \in \mathcal{S}(G)$ has the SAP. The reason behind this is because a 3-cycle appears in $\overline{G}$.

As shown in Example 2.1 and Example 2.2, some graph structures guarantee that every matrix described by the graph has the SAP. This assurance is given by forcing $x_e = 0$ step by step or by the occurrence of some odd cycle inside $\overline{G}$. Utilizing these ideas, we design the *SAP zero forcing game*, where the information $x_e = 0$ is stored by coloring the non-edge e blue.

Different from the conventional zero forcing game, the SAP zero forcing game is coloring "non-edges" to be blue or white, instead of coloring vertices; also, a set of initial blue non-edges is called a zero forcing set if every non-edge turns blue eventually by repeated applications of the given color change rules.

Let G be a graph and $i \in V(G)$. Recall that $N_G(i)$ is the neighborhood of i in G. For B_E a set of edges (2-sets), by considering $\langle B_E \rangle$ as the graph with its edge set B_E on the required vertices, the notation $N_{\langle B_E \rangle}(i)$ denotes the vertices j with $\{i, j\} \in B_E$.

The definition of $Z_{\text{SAP}}(G)$ uses the concept of local games, which we now define.

Definition 2.3. Let G be a graph with some non-edges B_E colored blue, and $k \in V(G)$. The *local game* $\phi_Z(G, B_E, k)$ is the conventional zero forcing game on G equipped with CCR-Z and the initial blue set $\phi_k(G, B_E) := N_G[k] \cup N_{\langle B_E \rangle}(k)$. When Z is replaced by another zero forcing rules, such as Z_ℓ or Z_+, the setting remains the same but a different rule is applied.

Definition 2.4. For a graph G, the *SAP zero forcing number* $Z_{\text{SAP}}(G)$ is the minimum number of blue non-edges such that every non-edge will become blue by repeated applications of the *color change rule for* Z_{SAP} (CCR-Z_{SAP}):

- Suppose at some stage, B_E is the set of blue non-edges and $\{j, k\}$ is a white non-edge. If $i \to j$ in $\phi_Z(G, B_E, k)$ for some vertex i, then the non-edge $\{j, k\}$ may turn blue. This is denoted as $(k : i \to j)$.
- Let $\overline{G}_W$ be the graph whose edges are the white non-edges. If for some vertex i, $\overline{G}_W[N_G(i)]$ contains a component that is an odd cycle C, then all edges in $E(C)$ may turn blue. This is denoted as $(i \to C)$.

The three vertices i, j, and k in the first rule are called a *forcing triple*; the second rule is called the *odd cycle rule*.

Note that a complete graph $G = K_n$ is considered as having all non-edges blue initially, so $Z_{\text{SAP}}(G) = 0$. The odd cycle rule follows a similar idea from the odd cycle zero forcing number [20].

Lemma 2.5. *For any nonzero real numbers $a_1, a_2, \ldots, a_n$ with n odd, a matrix of the form*

$$\begin{bmatrix} a_2 & 0 & \cdots & 0 & a_n \\ a_1 & a_3 & 0 & & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & a_n & 0 \\ 0 & \cdots & 0 & a_{n-1} & a_1 \end{bmatrix}$$

is nonsingular.

Proof. Let A be a matrix of the described form. When n is odd,

$$\det(A) = 2\prod_{i=1}^{n} a_i,$$

which is nonzero provided that a_i's are all nonzero. Hence A is nonsingular. □

Theorem 2.6. *Suppose G is a graph with $Z_{\mathrm{SAP}}(G) = 0$. Then every matrix in $\mathcal{S}(G)$ has the SAP. Therefore, $M(G) = \xi(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$.*

Proof. Let $A = [\, a_{i,j} \,] \in \mathcal{S}(G)$ with $\mathbf{v}_j$ as the j-th column vector. Pick an order for the set of non-edges, and let Ψ be the SAP matrix for A with respect to the given order. Suppose $\mathbf{x}$ is a vector such that $\Psi\mathbf{x} = \mathbf{0}$. Then $\mathbf{x} = (x_e)_{e\in E(\overline{G})}$ such that the entries of $\mathbf{x}$ are indexed by the non-edges of G in the given order. We relate the SAP zero forcing game to the zero-nonzero pattern of $\mathbf{x}$.

Claim 1: Suppose at some stage, B_E is the set of blue non-edges, and $(k : i \to j)$ is a forcing triple. Then $x_e = 0$ for all $e \in B_E$ implies $x_{\{j,k\}} = 0$.

To establish the claim, recall that the condition $\Psi\mathbf{x} = \mathbf{0}$ on those rows in the k-th block means

$$\sum_{s\notin N_G[k]} x_{\{s,k\}}\mathbf{v}_s = \mathbf{0}.$$

Suppose $x_e = 0$ for all $e \in B_E$. Then this equality reduces to

$$\sum_{s\notin N_G[k]\cup N_{\langle B_E\rangle}(k)} x_{\{s,k\}}\mathbf{v}_s = \mathbf{0}.$$

Since by Definition 2.3 the set $\phi_k(G, B_E) = N_G[k] \cup N_{\langle B_E\rangle}(k)$ is exactly the set of initial blue vertices in $\phi_Z(G, B_E, k)$, the force $i \to j$ in $\phi_Z(G, B_E, k)$ implies $x_{\{j,k\}} = 0$ by Proposition 1.4.

Claim 2: Suppose at some stage, B_E is the set of blue non-edges, and $(i \to C)$ is applied by the odd cycle rule. Then $x_e = 0$ for all $e \in B_E$ implies $x_e = 0$ for every $e \in E(C)$.

To establish the claim, let $\overline{G}_W$ be the graph whose edges are the white non-edges at this stage. Since $(i \to C)$ is applied by the odd cycle rule, C is a component in $\overline{G}_W[N_G(i)]$ and $|V(C)| = d$ is an odd number. Following the cycle order, write the vertices in $V(C)$ as $\{k_s\}_{s=1}^d$, and $e_s = \{k_s, k_{s+1}\}$, with the index taken modulo d.

Denote $U = \{(i, k_s)\}_{s=1}^d$, $W_1 = \{e_s\}_{s=1}^d$, and W_2 as those white non-edges not in W_1. Now $\{B_E, W_1, W_2\}$ forms a partition of $E(\overline{G})$. We have no control about $\Psi[U, B_E]$, but will show $\Psi[U, W_1]$ is always nonsingular and $\Psi[U, W_2] = O$. Consequently, $x_e = 0$ for all $e \in B_E$ implies $x_e = 0$ for every non-edge $e \in W_1 = E(C)$.

For each $(i, k_s) \in U$, $\Psi_{(i,k_s),e_{s-1}} = a_{i,k_{s-1}}$ and $\Psi_{(i,k_s),e_s} = a_{i,k_{s+1}}$, while both of them are nonzero; at the same time, $\Psi_{(i,k_s),e} = 0$ for all $e \in W_1$ other than e_{s-1} and e_s, since e is not incident to k_s. Therefore, $\Psi[U, W_1]$ is of the form described in Lemma 2.5, and it must be nonsingular.

On the other hand, consider a non-edge $\{j, h\} \in W_2$ and $(i, k_s) \in U$. If $k_s \notin \{j, h\}$, then $\Psi_{(i,k_s),\{j,h\}} = 0$. If $k_s \in \{j, h\}$, say $k_s = h$, then $j \notin N_G(i)$ (for otherwise k_s has degree at least 3 in $\overline{G}_W[N_G(i)]$ and the component containing k_s cannot be an odd cycle); this means $\{i, j\} \notin E(G)$ and $\Psi_{(i,k_s),\{j,h\}} = a_{i,j} = 0$. Therefore, $\Psi[U, W_2] = O$.

By the claims, $Z_{\mathrm{SAP}}(G) = 0$ means all of the x_e will be forced to zero, so $\mathbf{x} = \mathbf{0}$ is the only vector in the right kernel of Ψ. This means Ψ is full-rank.

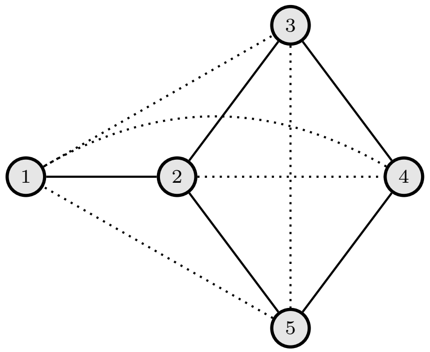
Since the argument works for every matrix $A \in \mathcal{S}(G)$, $Z_{\mathrm{SAP}}(G) = 0$ implies every matrix $A \in \mathcal{S}(G)$ has the SAP. Consequently, $M(G) = \xi(G)$, $M_+(G) = \nu(G)$, and $M_\mu(G) = \mu(G)$. □

Remark 2.7. With and without the restriction of having the SAP, the inertia sets that can be achieved by matrices in $\mathcal{S}(G)$ are considered in the literature (e.g., see [1,6]). With the help of Theorem 2.6, if $Z_{\mathrm{SAP}}(G) = 0$, then these two inertia sets are the same.

Corollary 2.8. *If G has no isolated vertices and $\overline{G}$ is a forest, then $Z_{\mathrm{SAP}}(G) = 0$ and every matrix in $\mathcal{S}(G)$ has the SAP.*

Proof. Suppose at some stage $\overline{G}_W$ is the graph whose edges are the white non-edges. Since $\overline{G}$ is a forest, $\overline{G}_W$ always has a leaf k, unless $\overline{G}_W$ contains no edges. Let j be the only neighbor of k in $\overline{G}_W$, and let i be one of the neighbors of j in G. Since G has no isolated vertices, i always exists. Thus, in the local game $\phi_Z(G, E(\overline{G}) \setminus E(\overline{G}_W), k)$, every vertex is blue except j, so $i \to j$. Therefore, $(k : i \to j)$ can be applied and $\{j, k\}$ turns blue. Continuing this process, all non-edges become blue, so $Z_{\mathrm{SAP}}(G) = 0$. □

Note that the condition that G has no isolated vertices is crucial for Corollary 2.8. For example, $Z_{\mathrm{SAP}}(\overline{K_{1,n}}) > 0$ when $n \geq 1$. In fact, $Z_{\mathrm{SAP}}(G) = 0$ does not happen only when $\overline{G}$ is a forest. Example 2.9 gives a graph G such that $\overline{G}$ is not a forest and $Z_{\mathrm{SAP}}(G) = 0$. We will see in Table 1 that there are a considerable number of graphs having the property $Z_{\mathrm{SAP}}(G) = 0$.



Step	Forcing triple	Forced non-edge
1	$(2 : 3 \to 4)$	$\{2, 4\}$
2	$(4 : 2 \to 1)$	$\{4, 1\}$
3	$(5 : 4 \to 3)$	$\{5, 3\}$
4	$(3 : 2 \to 1)$	$\{3, 1\}$
5	$(5 : 2 \to 1)$	$\{5, 1\}$

Fig. 2. The graph G for Example 2.9 and the forcing process.

Example 2.9. Let G be the graph shown in Fig. 2. Following the steps listed in Fig. 2, every non-edge turns blue, so $Z_{\mathrm{SAP}}(G) = 0$. Observe that at the beginning, the graph $\overline{G}_W$ of white non-edges is the same as $\overline{G}$, and $\overline{G}_W[N_G(2)]$ is a 3-cycle C, so one can also use the odd cycle rule to perform $(2 \to C)$. This will accelerate the process but not change the result. By Theorem 2.6, every matrix $A \in \mathcal{S}(G)$ has the SAP, so $\xi(G) = M(G)$. Since the number of vertices is no more than 7, $M(G) = Z(G) = 2$ and thus $\xi(G) = 2$.

Similar to Corollary 2.8, Remark 2.10 also provides some intuition of the SAP zero forcing process.

Remark 2.10. Suppose at some stage B_E is the set of blue non-edges on a graph G. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. If for some vertex i, the induced subgraph $\overline{G}_W[N_G(i)]$ has a leaf k with its only neighbor j in $\overline{G}_W[N_G(i)]$, then $(k : i \to j)$ can be applied on G, because in $\phi_Z(G, B_E, k)$ every vertex in $N_G(i)$ is blue except j.

This means whenever $\overline{G}_W[N_G(i)]$ contains a component that is a tree, every non-edge in this tree can turn blue inductively by forcing triples. Consequently, if at some stage $(i \to C)$ can be applied but some non-edges from C turns blue because of some forcing triples or other odd cycle rules, all edges in $E(C)$ can still turn blue by forcing triples, but not by $(i \to C)$.

Corollary 2.11. *Let G be any graph with diameter 2 and maximum degree at most 3. Then $Z_{\mathrm{SAP}}(G) = 0$. In particular, when G is the Petersen graph, $Z_{\mathrm{SAP}}(G) = 0$, so $\xi(G) = M(G) = 5$.*

Proof. For every white non-edge $\{j, k\}$, there is at least one common neighbor i of j and k, since the diameter is 2. By the assumption, $\deg_G(i) \leq 3$. Since i has at least two neighbors, $\deg_G(i) \geq 2$. If $\deg_G(i) = 2$, then $(k : i \to j)$. Suppose $\deg_G(i) = 3$. On the set $N_G(i)$, the white non-edges can form P_2, P_3, or C_3. In the case of P_2 and P_3, all non-edges in $N_G(i)$ turn blue by the argument in Remark 2.10. If it is C_3, then apply the odd cycle rule ($i \to C$). Since this argument works for every white non-edge, all non-edges can turn blue. Hence $Z_{\rm SAP}(G) = 0$.

Let G be the Petersen graph. Then G is a 3-regular graph with diameter 2. Thus, $Z_{\rm SAP}(G) = 0$, and $\xi(G) = M(G)$ by Theorem 2.6. It is known [2] that $M(G) = 5$. $\square$

In [5], it is asked if $\xi(G) \leq \xi(G - v) + 1$ for every graphs G and every vertex v of G. Theorem 2.6 answers this question in positive for a large number of graph-vertex pairs.

Corollary 2.12. *Let G be a graph and $v \in V(G)$. Suppose $Z_{\rm SAP}(G - v) = 0$. Then $\xi(G) \leq \xi(G - v) + 1$.*

Proof. Since $Z_{\rm SAP}(G - v) = 0$, $\xi(G - v) = M(G - v)$ by Theorem 2.6. Therefore,

$$\xi(G) \leq M(G) \leq M(G - v) + 1 = \xi(G - v) + 1,$$

where the inequality $M(G) \leq M(G - v) + 1$ is given in [11]. $\square$

Example 2.13. Let G be one of the tetrahedron K_4, cube Q_3, octahedron G_8, dodecahedron G_{12}, or icosahedron G_{20}. Then, $Z_{\rm SAP}(G) = 0$. This is trivial for tetrahedron, since it is a complete graph. The complement of an octahedron is three disjoint edges, which is a forest, so $Z_{\rm SAP}(G) = 0$. For the other three graphs, pick one vertex i and look at its neighborhood $N_G(i)$. The induced subgraph of $\overline{G}$ on $N_G(i)$ is either a 3-cycle or a 5-cycle. Thus the odd cycle rule or the argument in Remark 2.10 could be applied, and every non-edge in $N_G(i)$ turns blue. After doing this to every vertex, by picking one vertex and look at its local game, all white non-edges incident to this vertex will turn blue. Therefore, $\xi(G) = M(G)$.

It is known [16] that $M(K_4) = 3$ and $M(Q_3) = 4$. Since the octahedron graph is strongly regular, in [2] it shows $4 \leq M(G_8)$; together with the fact $Z(G_8) \leq 4$, we know $M(G_8) = 4$. For G_{12} and G_{20}, the zero forcing numbers can be computed through the computer program and both equal to 6, but the maximum nullity is not yet known.

Definition 2.14. Let G be a graph with some non-edges B_E colored blue. The color change rule for $Z^+_{\rm SAP}$ (CCR-$Z^+_{\rm SAP}$) is the following:

- Let $\{j, k\}$ be a non-edge. If $i \to j$ in $\phi_{Z_+}(G, B_E, k)$ for some vertex i, then the non-edge $\{j, k\}$ may turn to blue. This is denoted as $(k : i \to j)$.
- The odd cycle rule can be used to perform a force.

Similarly, the color change rule of $Z^\ell_{\rm SAP}$ (CCR-$Z^\ell_{\rm SAP}$) is defined through the local game $\phi_{Z_\ell}(G, B_E, i)$. As usual, $Z^+_{\rm SAP}(G)$ (respectively, $Z^\ell_{\rm SAP}$) is the minimum number of blue non-edges such that every non-edge will become blue by repeated applications of CCR-$Z^+_{\rm SAP}$ (respectively, CCR-$Z^\ell_{\rm SAP}$).

Observation 2.15. For any graph G, $Z^{+}_{\mathrm{SAP}}(G) \leq Z^{\ell}_{\mathrm{SAP}}(G) \leq Z_{\mathrm{SAP}}(G)$.

By a proof analogous to that of Theorem 2.6, we can establish Theorem 2.16. Observe that $Z^{\ell}_{\mathrm{SAP}}(G) = 0$ implies $Z^{+}_{\mathrm{SAP}}(G) = 0$.

Theorem 2.16. *Let G be a graph. If $Z^{\ell}_{\mathrm{SAP}}(G) = 0$, then every matrix in $\mathcal{S}_{\ell}(G)$ has the SAP. If $Z^{+}_{\mathrm{SAP}}(G) = 0$, then every matrix in $\mathcal{S}_{+}(G)$ has the SAP. Therefore, if $Z^{+}_{\mathrm{SAP}}(G) = 0$, then $M_{+}(G) = \nu(G)$.*

Corollary 2.17. *Suppose G is a graph with $Z^{+}_{\mathrm{SAP}}(G) = 0$. Then $\xi(G) \geq M_{+}(G)$.*

Example 2.18. Let $G = K_{n_1,n_2,\ldots,n_p}$ be a complete multi-partite graph with $n_1 \geq n_2 \geq \cdots \geq n_p$ and $p \geq 2$. Denote $n = \sum_{t=1}^{p} n_t$. Then $Z^{\ell}_{\mathrm{SAP}}(G) = Z^{+}_{\mathrm{SAP}}(G) = 0$, so $\nu(G) = M_{+}(G) = n - n_1$ [12]. On the other hand, if $n_1 \geq 4$, then $Z_{\mathrm{SAP}}(G) > 0$, since none of the non-edges in this part can turn blue.

Example 2.19. If T is a tree, then $Z^{+}_{\mathrm{SAP}}(T) = 0$. However, not every tree T has $Z^{\ell}_{\mathrm{SAP}}(T) = 0$. For example, let G be the graph obtained from $K_{1,4}$ by attaching four leaves to the four existing leaves. In this graph, only the non-edges incident to the center vertex can turn blue by CCR-Z^{ℓ}_{SAP}, so $Z^{\ell}_{\mathrm{SAP}}(G) > 0$.

2.1. Graph join

Since the SAP zero forcing process uses a propagation on non-edges, it is interesting to consider $Z_{\mathrm{SAP}}(G)$ if $\overline{G}$ has two or more components; that is, G is a join of two or more graphs.

Proposition 2.20. *Let G and H be two graphs. Then*

$$Z_{\mathrm{SAP}}(G \vee H) = Z_{\mathrm{SAP}}(G \vee K_1) + Z_{\mathrm{SAP}}(H \vee K_1).$$

Proof. Let v be the vertex corresponding to the K_1 in $G \vee K_1$. Denote $E_1 = E(\overline{G})$ and $E_2 = E(\overline{H})$. Consider the mapping $\pi : V(G \vee H) \to V(G \vee K_1)$ such that $\pi(i) = i$ if $i \in V(G)$ and $\pi(i) = v$ if $i \in V(H)$. Fix a vertex $u \in V(H)$, consider the mapping $\pi^{-1} : V(G \vee K_1) \to V(G \vee H)$ such that $\pi^{-1}(i) = i$ if $i \in V(G)$ and $\pi^{-1}(v) = u$.

Suppose at some stage B_E is the set of blue non-edges in $G \vee H$, and $B_E \cap E_1$ and $B_E \cap E_2$ are the sets of blue non-edges in $G \vee K_1$ and $H \vee K_1$ respectively. Let $e = \{j, k\} \in E_1$. If $(k : i \to j)$ happens in $G \vee H$, then $(k : \pi(i) \to j)$ can be applied in $G \vee K_1$; if $(k : i \to j)$ happens in $G \vee K_1$, then $(k : \pi^{-1}(i) \to j)$ can be applied in $G \vee H$. Also, if e is in some cycle C and $(i \to C)$ happens in either $G \vee H$ or $G \vee K_1$, then by the definition of the odd cycle rule C must totally fall in $V(G)$. If $(i \to C)$ in $G \vee H$, then $(\pi(i) \to C)$ in $G \vee K_1$; if $(i \to C)$ in $G \vee K_1$, then $(\pi^{-1}(i) \to C)$ in $G \vee H$. Similarly, all these correspondences work when $e \in E_2$.

Therefore, we can conclude that B_E is a ZFS-Z_{SAP} in $G \vee H$ if and only if $B_E \cap E_1$ and $B_E \cap E_2$ are ZFS-Z_{SAP} in $G \vee K_1$ and $H \vee K_1$ respectively. $\square$

Example 2.21. The value of $Z_{\text{SAP}}(G \vee K_1)$ and the value of $Z_{\text{SAP}}(G)$ can vary a lot. For example, when $G = \overline{K_n}$, we will show that $Z_{\text{SAP}}(\overline{K_n}) = \binom{n}{2}$ and $Z_{\text{SAP}}(\overline{K_n} \vee K_1) = Z_{\text{SAP}}(K_{1,n}) = \binom{n-1}{2} - 1$ when $n \geq 3$.

Since there are no edges in $\overline{K_n}$, no vertices can make a force in any local game, and odd cycle rules cannot be applied, either. This means $Z_{\text{SAP}}(\overline{K_n}) = \binom{n}{2}$.

For $K_{1,n}$, color some edges B_E of $\overline{K_{1,n}}$ blue so that the set of white non-edges forms a 3-cycle with $n-3$ leaves attaching to a vertex of the 3-cycle. Then B_E is a ZFS-Z_{SAP} for $K_{1,n}$, since the $n-3$ leaves can turn blue by forcing triples, and then the 3-cycle can turn blue by the odd cycle rule. Therefore, $Z_{\text{SAP}}(K_{1,n}) \leq \binom{n-1}{2} - 1$.

The inequality $Z_{\text{SAP}}(K_{1,3}) \leq \binom{3-1}{2} - 1 = 0$ implies $Z_{\text{SAP}}(K_{1,3}) = 0$, so we may assume $n \geq 4$. Suppose B_E is a ZFS-Z_{SAP} of $K_{1,n}$ with $|B_E| = \binom{n-1}{2} - 2$. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. Then $|E(\overline{G}_W)| = n + 1$. Obtain a subgraph H of $\overline{G}_W$ by deleting leaves and isolated vertices repeatedly until there is no leaves or isolated vertices left. By the choice of H, it is either $|V(H)| = 0$ or H has the minimum degree at least two. Since deleting a leaf removes an edge and a vertex, $|V(H)| + 1 \leq |E(H)|$, implying $|E(H)| \neq 0$ and $|V(H)| \neq 0$. Now H is a graph with minimum degree at least two and $|V(H)| + 1 \leq |E(H)|$; therefore, H must contain a component that is not a cycle (so in particular not an odd cycle). Let $\{j, k\}$ be an edge in this component. If $(k : i \to j)$ force $\{j, k\}$ to turn blue for some i, then i must be the center vertex of $K_{1,n}$. However, in $\phi_Z(G, B_E, k)$, vertex i has at least two white neighbors, because k has degree at least two in H. Therefore, no edges in this component can turn blue by either a forcing triple or an odd cycle rule, a contradiction. Hence $Z_{\text{SAP}}(K_{1,n}) = \binom{n-1}{2} - 1$.

Proposition 2.22. *For any graph G, $Z_{\text{SAP}}(G \vee K_1) \leq Z_{\text{SAP}}(G)$. If G contains no isolated vertices, then $Z_{\text{SAP}}(G \vee K_1) = Z_{\text{SAP}}(G)$.*

Proof. Every ZFS-Z_{SAP} for G is a ZFS-Z_{SAP} for $G \vee K_1$, so $Z_{\text{SAP}}(G \vee K_1) \leq Z_{\text{SAP}}(G)$.

Now consider the case that G has no isolated vertices. Suppose at some stage B_E is the set of blue non-edges for both $G \vee K_1$ and G. We claim that if a non-edge $\{j, k\} \in E(\overline{G})$ turns blue in $G \vee K_1$, then it can also turn blue in G.

Label the vertex in $V(K_1)$ as v. If $(k : i \to j)$ in $G \vee K_1$ with $i \neq v$, then it is also a forcing triple in G. Suppose $(k : v \to j)$ happens in $G \vee K_1$. Then it must be the case when j is the only white vertex in $\phi_Z(G \vee K_1, B_E, k)$, since v is a vertex that is adjacent to every vertex and it cannot make a force unless every vertex except j is already blue. Since j is not an isolated vertex, it has a neighbor i' in $V(G)$. Now $(k : i' \to j)$ can force $\{j, k\}$ to turn blue, since j is also the only white vertex in $\phi_Z(G, B_E, k)$.

On the other hand, if $(i \to C)$ happens in $G \vee K_1$ with $i \neq v$, then it can also happen in G. Suppose $(v \to C)$. Then every vertex in C is incident to exactly two white non-edges by the odd cycle rule, since v is adjacent to every vertex. Label the vertices of C by $\{k_s\}_{s=1}^d$ in the cycle order, with the index taken modulo d. In the local game $\phi_Z(G, B_E, k_2)$, there are only two white vertices, namely k_1 and k_3. Since G has no isolated vertices, k_1 has a neighbor i' in $V(G)$. If i' is not adjacent to k_3, then $(k_2 : i' \to k_1)$ can be applied and then the argument in Remark 2.10 can force all edges in $E(C)$ to turn blue. Therefore, we may assume i' is adjacent to k_3. By applying the same argument to k_4, we know i' is also adjacent to k_5. Inductively, i' is adjacent to all vertices in C, since C is an odd cycle. Therefore, $(i' \to C)$ can happen in G.

In conclusion, $Z_{\mathrm{SAP}}(G \vee K_1) = Z_{\mathrm{SAP}}(G)$. □

Proposition 2.23. *Let G be a graph. Then $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ if and only if one of the following holds:*

- *G has no isolated vertices and $Z_{\mathrm{SAP}}(G) = 0$.*
- *$G = K_1$ or G is a disjoint union of a connected graph H and an isolated vertex such that $Z_{\mathrm{SAP}}(H) = 0$.*
- *$G = \overline{K_3}$.*

Proof. Let v be the vertex in $V(K_1) \subseteq V(G \vee K_1)$. In the case that G has no isolated vertices, $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ if and only if $Z_{\mathrm{SAP}}(G) = 0$ by Proposition 2.22. If $G = K_1$, then $Z_{\mathrm{SAP}}(K_2) = 0$. If $G = \overline{K_3}$, then $Z_{\mathrm{SAP}}(K_{1,3}) = 0$. Finally, suppose G is a disjoint union of a connected graph H and an isolated vertex w such that $Z_{\mathrm{SAP}}(H) = 0$. Then every forcing triple and every odd cycle rule in H can work in $G \vee K_1$, so all non-edges of $G \vee K_1$ that are in the part of H can turn blue. After that, $(k : v \to w)$ takes action in $G \vee K_1$ for every $k \in V(H)$. Thus, every non-edge in $G \vee K_1$ is blue.

For the converse statement, suppose $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ and no initial blue non-edge is given for $G \vee K_1$. Suppose G has p components with vertex sets $V_1, V_2, \dots, V_p$. Call a non-edge with two endpoints in different components in G as a crossing non-edge. We claim that if $p \geq 3$, then no crossing non-edge can turn blue in $G \vee K_1$ by any forcing triples. Let $\{j, k\}$ be a crossing non-edge. Without loss of generality, let $k \in V_1$ and $j \in V_2$. Suppose at some stage B_E is the set of blue non-edges and none of the crossing non-edges is blue. In the local game $\phi_Z(G \vee K_1, B_E, k)$, all blue vertices are contained in $V_1 \cup \{v\}$, since all the crossing non-edges are white. If $(k : i \to j)$ happens in $G \vee K_1$, it must be the case that $i = v$, since v is the only blue neighbor of j in $\phi_Z(G \vee K_1, B_E, k)$. Pick a vertex $u \in V_3$. Since both j and u are white neighbors of v in $\phi_Z(G \vee K_1, B_E, k)$, it is impossible that $(k : i \to j)$ is a forcing triple. In conclusion, if $Z_{\mathrm{SAP}}(G \vee K_1) = 0$ and G contains at least three components, then the odd cycle rule must be applied to the crossing non-edges. Therefore, G must be $\overline{K_3}$ in this case.

If G has only one component, then G contains no isolated vertices, unless $G = K_1$. Otherwise assume G has an isolated vertex and has exactly two components. Then G must be a disjoint union of a connected graph H and an isolated vertex w. Now we build a sequence of forces for H according to the forces in $G \vee K_1$. Suppose $(k : i \to j)$ happens in $G \vee K_1$ with $j, k \in V(H)$. If $i \in V(H)$, then $(k : i \to j)$ also works in H. If $i \notin V(H)$, then it must be $(k : v \to j)$. But v is adjacent to every vertex, so in $\phi_Z(G, B_E, k)$ every vertex except j must be blue. Since H is connected, there must be a vertex i' that is adjacent to j. Thus, $(k : i' \to j)$ can force $\{j, k\}$ to turn blue.

Suppose $(i \to C)$ for some i and odd cycle C. If $i \in V(H)$, then $V(C) \subset V(H)$ and $(i \to C)$ can be applied in H. Since i cannot be w, we assume $i = v$. If $w \in V(C)$, then $C - w$ forms a path and all edges on this path can turn blue in H, by the argument of Corollary 2.8.

Finally, we claim that $(v \to C)$ cannot happen in $G \vee K_1$ when $V(C) \subseteq V(H)$. For the purpose of obtaining a contradiction, suppose at some stage B_E is the set of blue non-edges and $(v \to C)$ happens. Let $\overline{G}_W$ be the graph whose edges are the white non-edges. Write $V(C) = \{k_s\}_{s=1}^{d}$ in the cycle order, with the index taken modulo d. Since C is a component in $\overline{G}_W[N_{G\vee K_1}(v)]$, every non-edge $\{k_s, w\}$ with $k_s \in V(C)$ is blue at this stage. When all non-edges $\{k_s, w\}$ were all white, no odd cycle rule can force any of them to turn blue, since w is incident to at least $d \geq 3$ white non-edges. Without loss of generality, assume that $\{k_1, w\}$ is the first non-edge to turn blue among $\{k_s, w\}$ with $k_s \in V(C)$. Only a forcing triple can possibly force $\{k_s, w\}$ to turn blue, and it must be $(k_1 : v \to w)$ or $(w : v \to k_1)$. Suppose this happened at the stage where B_{E_0} was the set of blue non-edges. It cannot be $(k_1 : v \to w)$ because v has at least three white neighbors k_2, k_d, and w in $\phi_Z(G \vee K_1, B_{E_0}, k_1)$; meanwhile, it cannot be $(w : v \to k_1)$, since v has at least $d \geq 3$ white neighbors in $\phi_Z(G \vee K_1, B_{E_0}, w)$. This yields a contradiction.

In conclusion, every possible force in $G \vee K_1$ corresponds to a force in H. Therefore, if $Z_{\mathrm{SAP}}(G \vee K_1) = 0$, then $Z_{\mathrm{SAP}}(H) = 0$. □

2.2. Computational results for small graphs

Table 1 shows the proportions of graphs that have certain parameters equal to 0, over all connected graphs with a fixed number of vertices. Graphs are not labeled and isomorphic graphs are considered as the same. The computation is done by *Sage* and the code can be found in [19].

Table 1
The proportion of graphs that satisfies $\zeta(G) = 0$, over all connected graphs on n vertices.

n	$Z_{\mathrm{SAP}} = 0$	$Z^{\ell}_{\mathrm{SAP}} = 0$	$Z^{+}_{\mathrm{SAP}} = 0$
1	1.0	1.0	1.0
2	1.0	1.0	1.0
3	1.0	1.0	1.0
4	1.0	1.0	1.0
5	0.86	0.95	0.95
6	0.79	0.92	0.92
7	0.74	0.89	0.89
8	0.73	0.88	0.88
9	0.76	0.89	0.89
10	0.79	0.90	0.91

In Section 4, we apply these results to help compute the value of $\xi(G)$ when $|V(G)| \leq 7$.

3. A vertex cover version of the SAP zero forcing game

As Example 2.21 points out, for a connected graph G on n vertices, the value of $Z_{\mathrm{SAP}}(G)$ can be much higher than n. This section considers a vertex cover version of the SAP zero forcing game. That is, if B is a set of vertices, then consider the *complementary closure* $\overline{cl}(B)$ as all those non-edges that are incident to any vertex in B. Now instead of picking some non-edges as blue at the beginning, we pick a set of vertices B, and color the set $\overline{cl}(B)$ blue initially.

Following this idea, a new parameter $Z_{\mathrm{vc}}(G)$ is defined with $0 \leq Z_{\mathrm{vc}}(G) \leq n$, and Theorem 3.2 shows that $M(G) - Z_{\mathrm{vc}}(G) \leq \xi(G)$.

Definition 3.1. For a graph G, the parameter $Z_{\mathrm{vc}}(G)$ is the minimum number of vertices B such that by coloring $\overline{cl}(B)$ blue, every non-edge will become blue by repeated applications of CCR-Z_{SAP} with the restriction

- $(k : i \to j)$ cannot perform a force if $i \in B$ and $\{i, k\} \in E(\overline{G})$.

A set $B \subseteq V(G)$ with this property is called a Z_{vc} zero forcing set.

Theorem 3.2. *Let G be a graph. Then*

$$M(G) - Z_{\mathrm{vc}}(G) \leq \xi(G).$$

Proof. For given G and $A = [\,a_{i,j}\,] \in \mathcal{S}(G)$, let $d = Z_{\mathrm{vc}}(G)$ and $\overline{m} = |E(\overline{G})|$. Pick an order for the set of non-edges, and let Ψ be the SAP matrix for A with respect to the given order. Let B be a ZFS-Z_{vc} with $|B| = d$. We will show that we can perturb the diagonal entries of A corresponding to B such that the new matrix has the SAP.

Let $W = E(\overline{G}) - \overline{cl}(B)$ be the initial white non-edges. Since B is a ZFS-Z_{vc}, every non-edge in W is forced to turn blue at some stage. Say at stage t, W_t is the set of white non-edges that are forced to turn blue. The set W_t can be one non-edge, or the edges of an odd cycle; thus, $\{W_t\}_{t=1}^{s}$ forms a partition of W, where s is the number of stages it takes to make all non-edges turn blue. Define U_t as follows: If W_t is a non-edge forced to turn blue by the forcing triple $(k : i \to j)$, then $U_t = \{(i, k)\}$; if W_t is a cycle forced by the odd cycle rule $(i \to C)$, then $U_t = \{(i, v)\}_{v \in V(C)}$. Let $U = \bigcup_{t=1}^{s} U_t$.

We first show that $\{U_t\}_{t=1}^{s}$ are mutually disjoint. Let $(i, k) \in U_{t_0}$ at some stage t_0. Suppose $(k : i \to j)$ happens at stage t_0. Right before the force, there must be exactly one white non-edge connecting k and $N_G(i)$, namely $\{j, k\}$, by CCR-Z. After the force, all white non-edges connecting k and $N_G(i)$ turn blue. Suppose $(i \to C)$ happens instead for some odd cycle C. Right before the force, there are exactly two white non-edges connecting k and $N_G(i)$, namely the two edges incident to k in C. After the force, all such white non-edges turn blue already. Therefore, (i,k) can appear in only one stage, and $\{U_t\}_{t=1}^s$ are mutually disjoint.

Next we show that $\Psi[U,W]$ is nonsingular. The proof of Theorem 2.6 shows that if W_{t_0} is given by the odd cycle rule for some step t_0, then $\Psi[U_{t_0},W_{t_0}]$ is nonsingular and $\Psi[U_{t_0},\bigcup_{t=t_0+1}^s W_t]=O$. We will see that the same property is also true when W_{t_0} a single non-edge. Suppose at stage t_0, the set of blue non-edges is B_E and $(k:i\to j)$ is applied. Thus, $U_{t_0}=\{(i,k)\}$ and $W_{t_0}=\{\{j,k\}\}$. By Definition 1.1,

$$\Psi[U_{t_0},W_{t_0}]=[\,\Psi_{(i,k),\{j,k\}}\,]=[\,a_{i,j}\,],$$

which is nonsingular, since $\{i,j\}$ is an edge. For any white non-edge e that is not incident to k, $\Psi_{(i,k),e}=0$. If $e=\{j',k\}$ is a white non-edge for some $j'\neq j$, then j' is not a neighbor of i, for otherwise i has two white neighbors in $\phi_Z(G,B_E,k)$; therefore, $\Psi_{(i,k),e}=a_{i,j'}=0$. By column/row permutations according to $\{W_t\}_{t=1}^d$ and $\{U_t\}_{t=1}^d$ respectively, $\Psi[U,W]$ becomes a lower triangular block matrix, with every diagonal block nonsingular. Hence $\Psi[U,W]$ is nonsingular.

Now give the non-edges in $\overline{cl}(B)$ an order. Following the order, for each non-edge $\{i,j\}$ in $\overline{cl}(B)$, put either (i,j) or (j,i) into another ordered set U_B. Since $\Psi_{(i,j),\{i,j\}}=a_{i,i}$, the diagonal entries of $\Psi[U_B,\overline{cl}(B)]$ are controlled by $a_{i,i}$ for some $i\in B$.

Consider the matrix

$$\Psi[U\cup U_B,W\cup\overline{cl}(B)]=\begin{bmatrix}\Psi[U,W] & \Psi[U,\overline{cl}(B)]\\ \Psi[U_B,W] & \Psi[U_B,\overline{cl}(B)]\end{bmatrix}.$$

We claim that those entry $a_{i,i}$ with $i\in B$ only appear on the diagonal of $\Psi[U_B,\overline{cl}(B)]$. For each $i\in B$, the only possible occurrence of $a_{i,i}$ is in the case $\Psi_{(i,k),\{i,k\}}=a_{i,i}$ for some vertex k and non-edge $\{i,k\}\in E(\overline{G})$. If $i\in B$ and $\{i,k\}\in E(\overline{G})$, then $\{i,k\}\in\overline{cl}(B)$. Therefore, $\Psi[U,W]$ and $\Psi[U_B,W]$ do not have this type of $a_{i,i}$ with $i\in B$ involved. Now it is enough to show $(i,k)\notin U$. Recall that $U=\bigcup_{t=1}^s U_t$. At stage t, if a forcing triple is applied, then $(i,k)\notin U_t$ since $(k:i\to j)$ is forbidden for any j by the definition; if the odd cycle rule is applied, then $(i,k)\notin U_t$ since $\{i,k\}\in E(\overline{G})$. Therefore, $\Psi[U,\overline{cl}(B)]$ contains no such $a_{i,i}$ with $i\in B$, either.

Let D_B be the diagonal matrix indexed by $V(G)$ with the i,i-entry 1 if $i\in B$ and 0 otherwise. Consider the matrix $A+xD_B$. By the discussion above, the SAP matrix of $A+xD_B$ contains the submatrix

$$\begin{bmatrix}\Psi[U,W] & \Psi[U,\overline{cl}(B)]\\ \Psi[U_B,W] & \Psi[U_B,\overline{cl}(B)]+xI\end{bmatrix}.$$

Since $\Psi[U,W]$ is nonsingular, the submatrix above is nonsingular when x is large enough.

This means, by changing $d = |B|$ diagonal entries of A, the corresponding SAP matrix becomes full-rank. Therefore,

$$M(G) - Z_{\mathrm{vc}}(G) \leq \mathrm{null}(A + xD_B) \leq \xi(G). \qquad \square$$

Remark 3.3. Theorem 3.2 actually proves that if B is a ZFS-Z_{vc}, then every matrix $A \in \mathcal{S}(G)$ attains the SAP by perturbing those diagonal entries corresponding to B.

In classical graph theory, a vertex cover of a graph G is a set of vertices S such that every edge in G is incident to some vertex in S; that is, $G - S$ contains no edges. The *vertex cover number* $\beta(G)$ is defined as the minimum cardinality of a vertex cover in the graph G. Corollary 3.4 below shows the relation between $M(G)$, $\xi(G)$, and $\beta(G)$.

Corollary 3.4. *Let G be a graph. Then*

$$M(G) - \beta(\overline{G}) \leq \xi(G).$$

Proof. Let S be a vertex cover of $\overline{G}$. Then S is a ZFS-Z_{vc}, since every non-edge is blue initially. Therefore, $Z_{\mathrm{vc}}(G) \leq \beta(G)$ and the desired inequality comes from Theorem 3.2. $\square$

Example 3.5. Let $G = K_3 \vee \overline{K_4}$. Then from the data in [9], $M(G) = Z(G) = 5$. Since G is a subgraph of $K_3 \vee P_4$, by minor monotonicity $\xi(G) \leq \xi(K_3 \vee P_4) \leq Z(K_3 \vee P_4) \leq 4$. On the other hand, by picking one of the vertex in $V(K_4)$, it forms a ZFS-Z_{vc}, since the initial white non-edges form a 3-cycle and the odd cycle rule applies. Thus $Z_{\mathrm{vc}}(G) = 1$ and $\xi(G) \geq M(G) - Z_{\mathrm{vc}}(G) = 4$. Therefore, $\xi(G) = 4$.

Notice that G contains a K_4 minor but not a K_5 minor, so we can only say $\xi(G) \geq \xi(K_4) = 3$ by considering K_p minors.

Similarly, we can define $Z^{\ell}_{\mathrm{vc}}(G)$ by changing CCR-Z_{SAP} to CCR-Z^{ℓ}_{SAP}. Then we have Theorem 3.6.

Theorem 3.6. *Let G be a graph. Then*

$$M_+(G) - Z^{\ell}_{\mathrm{vc}}(G) \leq \nu(G).$$

Remark 3.7. The proof of Theorem 3.2 relies on the fact $\Psi[U, W]$ is a lower triangular block matrix. This is not always true for Z_+. As a vertex can force two or more white vertices under CCR-Z_+, the sets $\{U_t\}_{t=1}^{s}$ might not be mutually disjoint and it is possible that $|U| < |W|$. Therefore, the same proof does not work for Z_+.

4. Values of $\xi(G)$ for small graphs

Analogous to $M(G) \leq Z(G)$, it is shown in [4] that $\xi(G) \leq \lfloor Z \rfloor(G)$, where $\lfloor Z \rfloor(G)$ is defined through a (conventional) zero forcing game with CCR-$\lfloor Z \rfloor$:

- CCR-Z can be used to perform a force. Or if i is blue, i has no white neighbors, and i was not used to make a force yet, then i may pick one white vertex j and force it to turn blue.

By using *Sage* and with the help of Theorem 2.6 and Theorem 3.2, we will see that $\lfloor Z \rfloor(G)$ agrees with $\xi(G)$ for graphs up to 7 vertices. This result also relies on some other lower bounds. The *Hadwiger number* $\eta(G)$ is defined as the largest p such that G has a K_p minor. Since $\xi(G)$ is minor monotone, it is known [4] that when $\eta(G) = p$

$$\xi(G) \geq \xi(K_p) = p - 1 = \eta(G) - 1.$$

The T_3-family is a family of six graphs [14, Fig. 2.1]. It is known [14] that a graph G contains a minor in the T_3-family if and only if $\xi(G) \geq 3$.

Lemma 4.1. *Let G be a connected graph with at most 7 vertices. Then at least one of the following is true:*

- $Z_{\mathrm{SAP}}(G) = 0$*, which implies* $\xi(G) = M(G)$.
- G *is a tree, which implies* $\xi(G) = 2$ *if* G *is not a path, and* $\xi(G) = 1$ *otherwise.*
- $\lfloor Z \rfloor(G) = M(G) - Z_{\mathrm{vc}}(G)$*, which implies* $\xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = \eta(G) - 1$*, which implies* $\xi(G) = \lfloor Z \rfloor(G)$.
- $\lfloor Z \rfloor(G) = 3$ *and* G *contains a* T_3*-family minor, which implies* $\xi(G) = 3$.

Proof. By running a *Sage* program [19], one of the five cases will happen. If $Z_{\mathrm{SAP}}(G) = 0$, then $\xi(G) = M(G)$ by Theorem 2.6. If G is a tree, then $\xi(G) \leq 2$, and the equality holds only when G is not a path [5]. Both $M(G) - Z_{\mathrm{vc}}(G)$ and $\eta(G) - 1$ are lower bounds of $\xi(G)$ by Theorem 3.2 and [4]. When one of the lower bounds meets with the upper bound $\lfloor Z \rfloor(G)$, $\xi(G) = \lfloor Z \rfloor(G)$. Finally, if G has a T_3-family minor, then $\xi(G) \geq 3$ [14]. In this case, $\xi(G) = 3$ when $\lfloor Z \rfloor(G) = 3$. $\square$

While $\xi(T) \leq 2$ for all tree T, the value of $\lfloor Z \rfloor(T)$ can be more than two. Example A.11. of [4] gives a tree T with $\lfloor Z \rfloor(T) = 3$; the graph T is shown in Fig. 3. However, $\xi(G) = \lfloor Z \rfloor(G)$ is still true when G is a tree and $|V(G)| \leq 7$.

Lemma 4.2. *Let G be a tree with at most 7 vertices. Then* $\xi(G) = \lfloor Z \rfloor(G)$.

Proof. When G is a tree, it is known [5] that $\xi(G) = 2$ when G is not a path, and $\xi(G) = 1$ if G is a path. When G is a path, then $\xi(G) = 1 = \lfloor Z \rfloor(G)$. Assume G is not

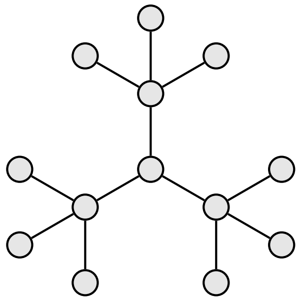


Fig. 3. An example of tree T with $\lfloor Z \rfloor(T) = 3$.

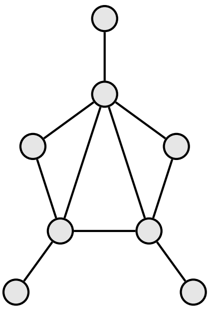


Fig. 4. A graph G on 8 vertices with $\xi(G) = 2$ but $\lfloor Z \rfloor(G) = 3$.

a path. It is enough to show $\lfloor Z \rfloor(G) \leq 2$. In this case, G must have a vertex v of degree at least 3. Call this type of vertex a high-degree vertex. If G has only one high degree vertex, then $\lfloor Z \rfloor(G) \leq 2$ since any two leaves form a ZFS-$\lfloor Z \rfloor$. Since $|V(G)| \leq 7$, there are at most two high-degree vertices. Pick two leaves such that the unique path between them contains only one high-degree vertex, then these two leaves form a ZFS-$\lfloor Z \rfloor$. $\square$

Theorem 4.3. *Let G be a graph with at most 7 vertices. Then $\xi(G) = \lfloor Z \rfloor(G)$.*

Proof. Let G be a graph with at most 7 vertices. Then $M(G) = Z(G)$ [9]. If $Z_{\mathrm{SAP}}(G) = 0$, then $\xi(G) = M(G) = Z(G)$. Since $\xi(G) \leq \lfloor Z \rfloor(G) \leq Z(G)$, $\xi(G) = \lfloor Z \rfloor(G)$. If G is a tree, then $\xi(G) = \lfloor Z \rfloor(G)$ by Lemma 4.2. Then by Lemma 4.1, $\xi(G) = \lfloor Z \rfloor(G)$ for all connected graph G up to 7 vertices. It is known that $\xi(G_1 \dot{\cup} G_2) = \max\{\xi(G_1), \xi(G_2)\}$ [5] and $\lfloor Z \rfloor(G_1 \dot{\cup} G_2) = \max\{\lfloor Z \rfloor(G_1), \lfloor Z \rfloor(G_2)\}$ [4], so $\xi(G) = \lfloor Z \rfloor(G)$ for any graph up to 7 vertices. $\square$

Example 4.4. Let G be the graph shown in Fig. 4. It is known [17] that $M(G) = 2$. Since G is not a disjoint union of paths, $\xi(G) = 2$. Also, it can be computed that $Z(G) = \lfloor Z \rfloor(G) = 3$.

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References

[1] M. Arav, F.J. Hall, Z. Li, H. van der Holst, The inertia set of a signed graph, Linear Algebra Appl. 439 (2013) 1506–1529.
[2] AIM Minimum Rank – Special Graphs Work Group, F. Barioli, W. Barrett, S. Butler, S.M. Cioabă, D. Cvetković, S.M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness, Zero forcing sets and the minimum rank of graphs, Linear Algebra Appl. 428 (2008) 1628–1648.
[3] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst, Zero forcing parameters and minimum rank problems, Linear Algebra Appl. 433 (2010) 401–411.
[4] F. Barioli, W. Barrett, S.M. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst, Parameters related to tree-width, zero forcing, and maximum nullity of a graph, J. Graph Theory 72 (2013) 146–177.
[5] F. Barioli, S.M. Fallat, L. Hogben, A variant on the graph parameters of Colin de Verdière: implications to the minimum rank of graphs, Electron. J. Linear Algebra 13 (2005) 387–404.
[6] W. Barrett, H.T. Hall, R. Loewy, The inverse inertia problem for graphs: cut vertices, trees, and a counterexample, Linear Algebra Appl. 431 (2009) 1147–1191.
[7] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in: Graph Structure Theory, American Mathematical Society, Providence, RI, 1993, pp. 137–147.
[8] Y. Colin de Verdière, Multiplicities of eigenvalues and tree-width graphs, J. Combin. Theory Ser. B 74 (1998) 121–146.
[9] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, G. Tims, Techniques for determining the minimum rank of a small graph, Linear Algebra Appl. 432 (2010) 2995–3001.
[10] R. Diestel, Graph Theory, 4th edition, Springer-Verlag, Heidelberg, 2010. Electronic edition, 2012.
[11] C.J. Edholm, L. Hogben, M. Huynh, J. LaGrange, D.D. Row, Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph, Linear Algebra Appl. 436 (2012) 4352–4372.
[12] S.M. Fallat, L. Hogben, Minimum rank, maximum nullity, and zero forcing number of graphs, in: L. Hogben (Ed.), Handbook of Linear Algebra, 2nd edition, CRC Press, Boca Raton, 2013.
[13] L. Hogben, Minimum rank problems, Linear Algebra Appl. 432 (2010) 1961–1974.
[14] L. Hogben, H. van der Holst, Forbidden minors for the class of graphs G with $\xi(G) \leq 2$, Linear Algebra Appl. 423 (2007) 42–52.
[15] H. van der Holst, A short proof of the planarity characterization of Colin de Verdière, J. Combin. Theory Ser. B 65 (1995) 269–272.
[16] L.-H. Huang, G.J. Chang, H.-G. Yeh, On minimum rank and zero forcing sets of a graph, Linear Algebra Appl. 432 (2010) 2961–2973.
[17] C.R. Johnson, R. Loewy, P.A. Smith, The graphs for which the maximum multiplicity of an eigenvalue is two, Linear Multilinear Algebra 57 (2009) 713–736.
[18] A. Kotlov, L. Lovász, S. Vempala, The Colin de Verdière number and sphere representations of a graph, Combinatorica 17 (1997) 483–521.
[19] J.C.-H. Lin, *Sage* code for Z_{SAP} related parameters, Published on the Iowa State *Sage* server at https://sage.math.iastate.edu/home/pub/54/. *Sage* worksheet available at https://github.com/jephianlin/publish/raw/master/VariationsOfZsap.sws.
[20] J.C.-H. Lin, Odd cycle zero forcing parameters and the minimum rank of graph blowups, Electron. J. Linear Algebra 31 (2016) 42–59.
[21] A. Schrijver, B. Sevenster, The Strong Arnold Property for 4-connected flat graphs, http://arxiv.org/abs/1512.03200.