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Zero forcing propagation time on oriented graphs

Adam Berliner [a], Chassidy Bozeman [b], Steve Butler [b,*], Minerva Catral [c], Leslie Hogben [b,d], Brenda Kroschel [e], Jephian C.-H. Lin [b], Nathan Warnberg [f], Michael Young [b]

[a] *Department of Mathematics, Statistics, and Computer Science, St. Olaf College, Northfield, MN 55057, USA*
[b] *Department of Mathematics, Iowa State University, Ames, IA 50011, USA*
[c] *Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207, USA*
[d] *American Institute of Mathematics, 600 E. Brokaw Road, San Jose, CA 95112, USA*
[e] *Department of Mathematics, University of St. Thomas, St. Paul, MN 55105, USA*
[f] *Department of Mathematics, University of Wisconsin-La Crosse, La Crosse, WI 54601, USA*

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A B S T R A C T

Zero forcing is an iterative coloring procedure on a graph that starts by initially coloring vertices white and blue and then repeatedly applies the following rule: if any blue vertex has a unique (out-)neighbor that is colored white, then that neighbor is forced to change color from white to blue. An initial set of blue vertices that can force the entire graph to blue is called a zero forcing set. In this paper we consider the minimum number of iterations needed for this color change rule to color all of the vertices blue, also known as the propagation time, for oriented graphs. We produce oriented graphs with both high and low propagation times, consider the possible propagation times for the orientations of a fixed graph, and look at balancing the size of a zero forcing set and the propagation time.

1. Introduction

Given a directed graph with no loops or multiple arcs (i.e., a simple digraph), there are many possible processes that can be used to simulate information spreading. In the simplest model, each vertex can have two states, knowing or not knowing (using the colors blue and white, respectively), and then have a color change rule for changing a vertex from not knowing to knowing (i.e., changes from white to blue). For each possible color change rule there are a variety of questions including finding the minimum number of vertices that if initially colored blue will eventually change all the vertices blue, or finding the length of time it takes for a graph to become blue. The goal in this paper is to consider a particular color change rule, known as zero forcing, and to focus on the amount of time it takes to turn all the vertices blue, known as propagation time, on digraphs and specifically oriented graphs.

The zero forcing process on a simple digraph is based on an initial coloring of each vertex as blue or white and the repeated application of the following coloring rule: If a blue vertex has exactly one white out-neighbor, then that out-neighbor will change from white to blue. In terms of rumor spreading, this can be rephrased in the following way: "If I know a secret and all except one of my friends knows the same secret, then I will share that secret with my friend that doesn't know". The *zero forcing number* is the minimum number of vertices initially colored blue that can transform the entire graph to blue.

* Corresponding author.
E-mail addresses: berliner@stolaf.edu (A. Berliner), cbozeman@iastate.edu (C. Bozeman), butler@iastate.edu (S. Butler), catralm@xavier.edu (M. Catral), hogben@aimath.org, hogben@iastate.edu (L. Hogben), bkkroschel@stthomas.edu (B. Kroschel), chlin@iastate.edu (J.C.-H. Lin), nwarnberg@uwlax.edu (N. Warnberg), myoung@iastate.edu (M. Young).

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The process of zero forcing was introduced originally for (simple, undirected) graphs by mathematical physicists in [8] and by combinatorial matrix theorists in [2]. A set of vertices that can color the entire graph blue can also control the quantum system [7]. The maximum nullity of a graph is the maximum of the nullities of symmetric matrices whose nonzero off-diagonal pattern is described by the edges of the graph. The zero forcing number of the graph is an upper bound on nullity of such a matrix [2] (in fact, the term "zero forcing" comes from forcing zeros in a null vector of such a matrix).

In general, the zero forcing number can be determined computationally but the problem of computing the zero forcing number is NP-hard [1]. However, the zero forcing number has been determined for several families of graphs (see, for example, the recent survey by Fallat and Hogben [12]), and bounds have been established in some cases [3,13,17]. There has also been additional work done on applications to control of quantum systems [7] and structural control [19]. Variants such as the k-forcing number [3] and fractional zero forcing [16] have recently been introduced and studied. Zero forcing was extended to digraphs in Barioli et al. [4]. Zero forcing for simple digraphs was studied in [5,14].

Most of the focus of the literature has been on the determination of the zero forcing number. However, given a minimum set of vertices initially colored blue that can transform the entire graph to blue, another natural question to examine is the amount of time it takes to turn all of the vertices blue (i.e., the propagation time). This study was initiated for undirected graphs in Hogben et al. [15] where extremal configurations were determined (i.e., n-vertex graphs that propagate as quickly or as slowly as possible) and in Chilakamarri et al. [11] where propagation time (there called *iteration index*) was computed for some families of graphs. This paper expands the study of propagation time to oriented graphs. In particular, there are some subtle and important distinctions between undirected graphs and oriented graphs.

In the remainder of the introduction we introduce the notation and give precise terminology. In Section 2 we show that the propagation time is not affected when the direction of each arc in a simple digraph is reversed. In Sections 3 and 4 we consider orientations of graphs that have low and high propagation times, respectively. For a given graph G there are many possible orientations and this gives rise to the following problem: *For a given graph G, find the propagation time of $\overrightarrow{G}$ as $\overrightarrow{G}$ ranges over all possible orientations of G.* In Section 5 we consider such orientation propagation intervals. In Section 6 we consider what happens when we balance the size of the zero forcing set with the propagation time; in particular we show that, unlike in simple graphs, we cannot always obtain significant savings in the sum by increasing the size of the zero forcing set. Finally, in Section 7 we discuss other approaches to propagation time for oriented graphs, including computation of propagation time for a given oriented graph and consideration of variation of propagation times across more than one minimum zero forcing set.

1.1. Terminology and definitions

A *simple graph* (respectively, *simple digraph*) is a finite undirected (respectively, directed) graph that does not allow loops or more than one copy of one edge or arc; a simple digraph does allow *double* arcs, i.e., both the arcs (u, v) and (v, u). We use $G = (V(G), E(G))$ to denote a simple graph and $\Gamma = (V(\Gamma), E(\Gamma))$ to denote a simple digraph, where V and E are the vertex and edge (or arc) sets, respectively. Furthermore, we let $|G| = |V(G)|$ denote the number of vertices of G, and similar notation is used for digraphs. An *oriented graph* is a simple digraph in which there are no double arcs, i.e., if (u, v) is an arc in Γ then (v, u) is not an arc in Γ. For a simple graph G, we also let $\overrightarrow{G}$ denote an orientation of G,[1] i.e., $\overrightarrow{G}$ is an oriented graph such that ignoring the orientations of the arcs gives the graph G.

For a digraph Γ having $u, v \in V(\Gamma)$ and $(u, v) \in E(\Gamma)$, we say that v is an *out-neighbor* of u and that u is an *in-neighbor* of v. The set of all in-neighbors of v is denoted by $N^-(v)$ and the cardinality of $N^-(v)$ is the *in-degree* of v, denoted by $\deg^-(v)$. Similarly, the set of all out-neighbors of v is $N^+(v)$ and the cardinality of $N^+(v)$ is the *out-degree*, denoted by $\deg^+(v)$.

For a simple digraph Γ, the zero forcing propagation process can be described as follows. Let $B \subseteq V(\Gamma)$, let $B^{(0)} := B$ and iteratively define $B^{(t+1)}$ as the set of vertices w where for some $v \in \bigcup_{i=0}^{t} B^{(i)}$ we have that w is the unique out-neighbor of v that is not in $\bigcup_{i=0}^{t} B^{(i)}$. Here $B^{(0)}$ represents the initial set of vertices colored blue, and at each stage we color as many vertices blue as possible (i.e., we apply the coloring rule simultaneously to all vertices). We say a set B is a *zero forcing set* if $\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)$ for some t. Further, the propagation time of B, denoted by $\mathrm{pt}(\Gamma, B)$, is the minimum t so that $\bigcup_{i=0}^{t} B^{(i)} = V(\Gamma)$ (i.e., the minimum amount of time needed for B to color the entire graph blue).

One way to achieve fast propagation is to simply let $B = V(\Gamma)$, and be done at time 0. However, we are primarily interested in the propagation time of a zero forcing set B of minimum cardinality, called a *minimum zero forcing set*. In particular, for a simple digraph Γ, we let $Z(\Gamma)$ denote the cardinality of a minimum zero forcing set for Γ. We then define propagation time as follows:

$$\mathrm{pt}(\Gamma) = \min\{\mathrm{pt}(\Gamma, B) : B \text{ is a minimum zero forcing set}\}.$$

Example 1.1. Consider the oriented graph $\overrightarrow{G}$ shown in Fig. 1. Since the vertices c, d and f are not the out-neighbors of any vertices, they cannot be changed to blue by the coloring rule. Therefore these three vertices *must* be in every zero forcing set of $\overrightarrow{G}$. We now show that these three vertices form a zero forcing set (and in particular this is the unique minimum cardinality zero forcing set), allowing us to conclude $Z(\overrightarrow{G}) = 3$. Suppose that $B^{(0)} = \{c, d, f\}$ and mark these vertices by

[1] For visual simplicity, the arrow is only over the main symbol, e.g., an orientation of K_n is denoted by $\overrightarrow{K}_n$ rather than $\overrightarrow{K_n}$.

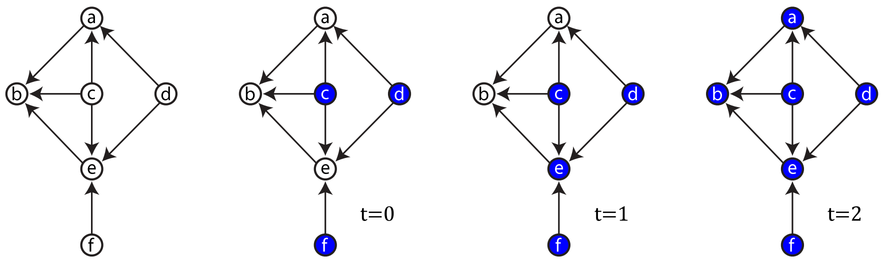

Fig. 1. An example of propagation for the zero forcing process.

coloring them blue (see $t = 0$ in Fig. 1). Since e is the unique white out-neighbor of f, we can color e blue. But this is the only vertex that can be colored at this time, and so we have $B^{(1)} = \{e\}$. The state of our coloring at time $t = 1$ is shown in Fig. 1. Now b is the unique white out-neighbor of e and a is the unique white out-neighbor of d, and so we can color both of them and we have $B^{(2)} = \{a, b\}$. At this stage all the vertices are blue and so the propagation time corresponding to this set is 2. As already noted, $\{c, d, f\}$ is the unique minimum zero forcing set, so $\text{pt}(\overrightarrow{G}) = 2$.

The oriented graph $\overrightarrow{G}$ in Example 1.1 has a unique minimum zero forcing set $B = \{c, d, f\}$, so $\text{pt}(\overrightarrow{G}) = \text{pt}(\overrightarrow{G}, B)$. However, it is often the case that an oriented graph has more than one minimum zero forcing set, and the propagation time of minimum zero forcing sets may vary. This topic is discussed further in Section 7.

Consider the zero forcing propagation process for a simple digraph Γ and zero forcing set B. When a white vertex v is the unique white out-neighbor of a blue vertex u, then we say that *u forces v* to change its color, and we write $u \to v$. Given a set B we can consider the set of arcs that correspond to forces that were used in coloring the graph. This collection of arcs is known as a *set of forces*, and denoted by $\mathcal{F}$. When there is a white vertex that could be changed to blue by two different in-neighbors we put only one of the corresponding arcs in $\mathcal{F}$. In particular, for a given set B there are possibly many different sets of forces for the propagation process. However, whether or not B is a zero forcing set, and similarly the propagation time, is not dependent on which choices made when including forcing arcs (see [6] and [15] for more information).

The subdigraph $(V, \mathcal{F})$ of $\Gamma = (V, E)$ is a collection of disjoint directed paths, where each vertex in B is the tail of a path. In particular, at each time in the propagation process, at most one vertex is added to each path, and thus $|B^{(t)}| \le |B|$ for all $t \ge 0$. The next observation is an immediate consequence.

Observation 1.2. *For a simple digraph Γ,*

$$\frac{|\Gamma| - Z(\Gamma)}{Z(\Gamma)} \le \text{pt}(\Gamma) \le |\Gamma| - Z(\Gamma).$$

Without loss of generality we can assume that our digraphs are connected (meaning the underlying simple graph is connected). This is because once the zero forcing numbers and propagation times on each component are known, the zero forcing number and propagation time on the whole graph are known. This is summarized in the next observation (the statement about zero forcing number appears in the literature, e.g., [6]).

Observation 1.3. *For a simple digraph Γ with connected components $\Gamma_1, \ldots, \Gamma_h$,*

$$Z(\Gamma) = \sum_{i=1}^{h} Z(\Gamma_i) \text{ and } \text{pt}(\Gamma) = \max_i \text{pt}(\Gamma_i).$$

A *Hessenberg path* is a simple digraph with vertices $\{v_1, v_2, \ldots, v_n\}$ that contains the arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$, and does not contain any arc of the form (v_i, v_j) with $j > i + 1$. Note that no restrictions are placed on arcs of the form (v_i, v_j) with $i > j + 1$, i.e., *back arcs* are allowed. (A single isolated vertex is a Hessenberg path.)

2. Reversing arcs

Although our focus is on oriented graphs, the results in this section are true for simple digraphs, so we state them that way. Given a simple digraph Γ, we let Γ^T be the simple digraph where the direction of each arc has been reversed. (Note that the adjacency matrix of Γ^T is the transpose of the adjacency matrix of Γ, which motivates the notation.) Reversing the arcs will generally change what the zero forcing sets are and how they propagate. However, we show in Theorem 2.5 that $\text{pt}(\Gamma^T) = \text{pt}(\Gamma)$, following the arguments in [15].

Let Γ be a simple digraph, B a zero forcing set of Γ, and $\mathcal{F}$ a set of forces of B. The *terminus* of $\mathcal{F}$, denoted by $\text{Term}(\mathcal{F})$, is the set of vertices that do not perform a force in $\mathcal{F}$, i.e., these are the heads of the directed paths formed by $\mathcal{F}$ (note that if a vertex in B never forces, then it is both the tail and head on a path with no arcs). Let $\text{Rev}(\mathcal{F})$ correspond to the set of forces found by reversing the direction of each arc of $\mathcal{F}$. Note that $\mathcal{F} \subseteq E(\Gamma)$ and $\text{Rev}(\mathcal{F}) \subseteq E(\Gamma^T)$.

Proposition 2.1. [6] *Let Γ be a simple digraph, B a zero forcing set of Γ, and $\mathcal{F}$ a set of forces of B. Then* $\mathrm{Term}(\mathcal{F})$ *is a zero forcing set for Γ^T and* $\mathrm{Rev}(\mathcal{F})$ *is a set of forces. Hence $Z(\Gamma^T) = Z(\Gamma)$.*

We previously have defined propagation in terms of an initial set B, but we can also define propagation using the set of forces $\mathcal{F}$.

Definition 2.2. Let $\Gamma = (V, E)$ be a simple digraph and B a zero forcing set of Γ. For a set of forces $\mathcal{F}$ of B that colors all vertices, define $\mathcal{F}^{(0)} = B$. For $t \geq 0$, let $\mathcal{F}^{(t+1)}$ be the set of vertices w such that for some $v \in \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$, the arc (v, w) appears in $\mathcal{F}$, $w \notin \bigcup_{i=0}^{t} \mathcal{F}^{(i)}$, and w is the only out-neighbor of v not in $\bigcup_{i=0}^{t} \mathcal{F}^{(i)}$. (Note that the set $\mathcal{F}$ is a collection of arcs, while the sets $\mathcal{F}^{(i)}$ are collections of vertices.) The *propagation time of $\mathcal{F}$ in Γ*, denoted by $\mathrm{pt}(\Gamma, \mathcal{F})$, is the minimum t such that $\bigcup_{i=0}^{t} \mathcal{F}^{(i)} = V(\Gamma)$.

We now give a connection between the propagation time given by $\mathcal{F}$ in Γ and the propagation time given by $\mathrm{Rev}(\mathcal{F})$ in Γ^T.

Lemma 2.3. *Let $\Gamma = (V, E)$ be a simple digraph, B a minimum zero forcing set, $\mathcal{F}$ a set of forces of B, and $1 \leq t \leq \mathrm{pt}(\Gamma, \mathcal{F})$. If $(v, u) \in \mathcal{F}$ with $u \in \mathcal{F}^{(\mathrm{pt}(\Gamma,\mathcal{F})-t+1)}$, then $v \in \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$.*

Proof. By Proposition 2.1 we have $\mathrm{Term}(\mathcal{F})$ is a zero forcing set for Γ^T with forcing set $\mathrm{Rev}(\mathcal{F})$. We establish the result by induction on t. For $t = 1$, let $u \in \mathcal{F}^{(\mathrm{pt}(\Gamma,\mathcal{F}))}$. Then $u \in \mathrm{Term}(\mathcal{F}) = \mathrm{Rev}(\mathcal{F})^{(0)}$. If $x \neq v$ is an in-neighbor of u in Γ, then x cannot force in Γ since $u \in \mathcal{F}^{(\mathrm{pt}(\Gamma,\mathcal{F}))}$. So $x \in \mathrm{Term}(\mathcal{F}) = \mathrm{Rev}(\mathcal{F})^{(0)}$. Hence, v is the only white out-neighbor of u in $\mathrm{Rev}(\mathcal{F})$. So $v \in \mathrm{Rev}(\mathcal{F})^{(1)}$.

Assume that the claim is true for $1 \leq s \leq t$. Suppose $u \in \mathcal{F}^{(\mathrm{pt}(\Gamma,\mathcal{F})-(t+1)+1)}$. Then $v \to u$ at time $\mathrm{pt}(\Gamma, \mathcal{F}) - t$ in Γ, so u cannot perform a force in Γ until $\mathrm{pt}(\Gamma, \mathcal{F}) - t + 1$ or later. Thus $u \in \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$, by the induction hypothesis in the case that u performs a force, and since $u \in \mathrm{Rev}(\mathcal{F})^{(0)}$ in the case that u does not perform a force. If $x \neq v$ is an in-neighbor of u in $\mathcal{F}$, then x cannot perform a force in Γ until $\mathrm{pt}(\Gamma, \mathcal{F}) - t + 1$ or later. So $x \in \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$. Thus, if $v \notin \bigcup_{i=0}^{t} \mathrm{Rev}(\mathcal{F})^{(i)}$, then $v \in \mathrm{Rev}(\mathcal{F})^{(t+1)}$, i.e., $v \in \bigcup_{i=0}^{t+1} \mathrm{Rev}(\mathcal{F})^{(i)}$ as desired. □

Since $\mathrm{Rev}(\mathrm{Rev}(\mathcal{F})) = \mathcal{F}$ and $(\Gamma^T)^T = \Gamma$, Lemma 2.3 implies the next result.

Corollary 2.4. *Let $\Gamma = (V, E)$ be a simple digraph, B a minimum zero forcing set of Γ, and $\mathcal{F}$ a forcing set of B. Then $\mathrm{pt}(\Gamma^T, \mathrm{Rev}(\mathcal{F})) = \mathrm{pt}(\Gamma, \mathcal{F})$.*

A minimum zero forcing set B of Γ is said to be an *efficient* zero forcing set if $\mathrm{pt}(\Gamma, B) = \mathrm{pt}(\Gamma)$. A set of forces $\mathcal{F}$ of an efficient forcing set B is *efficient* if $\mathrm{pt}(\Gamma, \mathcal{F}) = \mathrm{pt}(\Gamma)$.

Theorem 2.5. *Let $\Gamma = (V, E)$ be a simple digraph. Then $\mathrm{pt}(\Gamma^T) = \mathrm{pt}(\Gamma)$.*

Proof. Choose an efficient zero forcing set B and efficient set of forces $\mathcal{F}$, so $\mathrm{pt}(\Gamma, \mathcal{F}) = \mathrm{pt}(\Gamma)$. Then by Corollary 2.4,

$$\mathrm{pt}(\Gamma^T) \leq \mathrm{pt}(\Gamma^T, \mathrm{Rev}(\mathcal{F})) = \mathrm{pt}(\Gamma, \mathcal{F}) = \mathrm{pt}(\Gamma).$$

By reversing the roles of Γ and Γ^T, we also obtain the reverse inequality. □

3. Orientations with low propagation times

The smallest possible propagation time is 0. It is easy to see that for a connected oriented graph $\vec{G}$ of order at least two, $Z(\vec{G}) \leq |\vec{G}| - 1$ and $\mathrm{pt}(\vec{G}) \geq 1$. Thus, the only oriented graphs having propagation time equal to 0 are graphs with no edges, i.e., graphs consisting only of isolated vertices.

We now consider graphs that have an orientation with propagation time one. For such orientation, every vertex is in the zero forcing set or colored in the first time step, so we have the following observation.

Observation 3.1. *For an oriented graph $\vec{G}$ with $\mathrm{pt}(\vec{G}) = 1$, $Z(\vec{G}) \geq \left\lceil \frac{|\vec{G}|}{2} \right\rceil$.*

Therefore, the orientation must have a large zero forcing number (though this is not sufficient). Graphs having an orientation with propagation time one are not easy to classify. Many graphs, including trees (see Theorem 3.3) and complete graphs of order at least six (see Theorem 3.6), have such orientations. However, this is not true for all graphs. As the next example shows, there is no orientation of K_4 with propagation time one.

Example 3.2. All possible orientations of K_4, up to relabeling, are shown in Fig. 2, taken from and labeled as in [18]. It is known that $Z(\vec{K}_4) = 1$ if $\vec{K}_4$ is a Hessenberg path (D149, with path $(1, 2, 3, 4)$), in which case $\mathrm{pt}(\vec{K}_4) = 3$; for other orientations $Z(\vec{K}_4) = 2$ (see [5]). For D115 the only zero forcing set of cardinality two is $B_1 = \{1, 3\}$ and $\mathrm{pt}(\mathrm{D115}, B_1) = 2$. For D129 there are three possible zero forcing sets of cardinality two, but they are all equivalent by symmetry to $B_2 = \{2, 3\}$, and $\mathrm{pt}(\mathrm{D129}, B_2) = 2$. Since D122 is the reverse of D129, $\mathrm{pt}(\mathrm{D122}) = 2$.

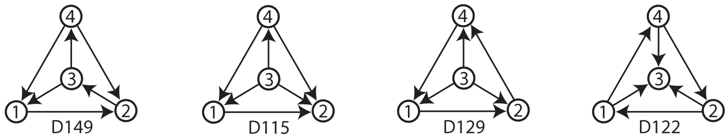

Fig. 2. Orientations of K_4.

3.1. Trees

In this section we show that any tree (and hence any forest) can be oriented to have propagation time one, unless it consists entirely of isolated vertices.

Theorem 3.3. *Let T be a tree on $n \geq 2$ vertices. Then there is an orientation $\overrightarrow{T}$ of T such that* $\mathrm{pt}(\overrightarrow{T}) = 1$.

Proof. A connected oriented graph $\overrightarrow{G}$ of order at least two has $\mathrm{pt}(\overrightarrow{G}) \geq 1$, so it is sufficient to show that any tree T of order at least two has an orientation $\overrightarrow{T}$ with $\mathrm{pt}(\overrightarrow{T}) \leq 1$. We prove this statement by induction.

For the base case, it can be seen that $\mathrm{pt}(\overrightarrow{T}) = 1$ when $n = 2$. Assume every nontrivial tree with fewer than n vertices can be oriented to have propagation time one and consider a tree T on n vertices. Choose a vertex y such that $\deg(y) \geq 2$ and at most one component of $T - y$ is a smaller tree T' of order two or more; any other components are isolated vertices which we denote by $z_1, z_2, \ldots, z_s$. If there is no component of order two or more, orient the edges of T as $z_i \to y$, for $i = 1, \ldots, s$, so $Z(\overrightarrow{T}) = n - 1$ and $\mathrm{pt}(\overrightarrow{T}) = 1$. Now assume there is a unique component T' of order at least two. By the induction hypothesis, there is an orientation $\overrightarrow{T}'$ of T' with $\mathrm{pt}(\overrightarrow{T}') = 1$. Let B be an efficient minimum zero forcing set of $\overrightarrow{T}'$, and let x denote the unique neighbor of y among $V(T')$.

First suppose $x \notin B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T}'$ by orienting edges of T not in T' so that $\deg^+(y) = 0$, i.e., $x \to y$ and $z_i \to y$, $1 \leq i \leq s$. Observe that $B \cup \{z_i\}_{i=1}^s$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T})$. Since $\deg^-(z_i) = 0$ ($1 \leq i \leq s$), any minimum zero forcing set of $\overrightarrow{T}$ must be of the form $\hat{B} \cup \{z_i\}_{i=1}^s$. In particular, $\hat{B}$ must force all vertices of $\overrightarrow{T}'$ without help from y or $\{z_i\}_{i=1}^s$, because y cannot contribute any forces to $V(T')$. Therefore, $|B| \leq |\hat{B}|$ and

$$Z(\overrightarrow{T}) \leq |B \cup \{z_i\}_{i=1}^s| \leq |\hat{B} \cup \{z_i\}_{i=1}^s| = Z(\overrightarrow{T}).$$

Next suppose $x \in B$. Obtain $\overrightarrow{T}$ from $\overrightarrow{T}'$ by orienting edges of T not in T' so that $\deg^-(y) = 0$, i.e., $y \to x$ and $y \to z_i$, for $i = 1, \ldots, s$. Observe that $B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$ is a zero forcing set of $\overrightarrow{T}$ with propagation time one. We show that $|B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| = Z(\overrightarrow{T})$. Since $\deg^-(y) = 0$ and y can force at most one of $\{z_i\}_{i=1}^s$ blue, y and at least $s - 1$ of the z_i must be blue initially. Thus without loss of generality, a minimum zero forcing set of $\overrightarrow{T}$ has the form $\hat{B} \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$. If $z_s \in \hat{B}$, then $(\hat{B} \setminus \{z_s\}) \cup \{x\} \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}$ is a zero forcing set with the same cardinality. Thus we can assume $x \in \hat{B}$, so $\hat{B}$ forces all vertices of $\overrightarrow{T}'$ without the help of $\{y\} \cup \{z_i\}_{i=1}^{s-1}$. Therefore, $|B| \leq |\hat{B}|$ and

$$Z(\overrightarrow{T}) \leq |B \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| \leq |\hat{B} \cup \{y\} \cup \{z_i\}_{i=1}^{s-1}| = Z(\overrightarrow{T}).$$

This completes the induction, and thus the proof. □

Corollary 3.4. *If T is a forest that contains an edge, then there is an orientation $\overrightarrow{T}$ of T such that* $\mathrm{pt}(\overrightarrow{T}) = 1$.

3.2. Tournaments

Trees are the sparsest connected graphs, i.e., those with the smallest possible number of edges. At the opposite extreme are tournaments, which are orientations of complete graphs. However, we will see that for all but two positive integers n there is a tournament on n vertices that has propagation time one. In particular, we see that minimum propagation time is not strongly correlated with density.

Proposition 3.5. *For $n \neq 2$, there is an orientation $\overrightarrow{K}_{2n}$ with* $\mathrm{pt}(\overrightarrow{K}_{2n}) = 1$.

Proof. Since the tournament of order 2 has propagation time one, we will assume $n \geq 3$. Let $\mathbb{Z}_n$ be the additive cyclic group of order n. We partition the vertices into two parts U and L, and index the vertices by $\mathbb{Z}_n$, in other words, $U := \{u_i : i \in \mathbb{Z}_n\}$ and $L := \{\ell_i : i \in \mathbb{Z}_n\}$. Place arcs between these vertices as follows: $A_1 = \{(u_i, \ell_{i-1}) : i \in \mathbb{Z}_n\}$ and $A_2 = \{(\ell_j, u_i) : j \neq i - 1\}$.

Also, define $A_3 = \{(u_i, u_j) : j - i \in W\}$, where $W = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \subseteq \mathbb{Z}_n$. If n is odd, then A_3 is properly defined. For $n = 2k$, each pair $\{u_i, u_{i+k}\}$ is doubly directed in A_3. In this case we modify A_3 by randomly choosing one of the arcs (u_i, u_{i+k}) or (u_{i+k}, u_i) for each pair $\{u_i, u_{i+k}\}$. Finally, we define $A_4 = \{(\ell_i, \ell_j) : (u_i, u_j) \in A_3\}$ and

$$E = A_1 \cup A_2 \cup A_3 \cup A_4.$$

Thus, $\overrightarrow{K}_{2n} = (V, E)$ is a tournament on $2n$ vertices, where $V = U \cup L$. For convenience, we call U the upper part of $\overrightarrow{K}_{2n}$ and L the lower part of $\overrightarrow{K}_{2n}$, and refer to forcing up or down.

Observe that U forms a zero forcing set of $\overrightarrow{K}_{2n}$, $|U| = n$, and $\mathrm{pt}(\overrightarrow{K}_{2n}, U) = 1$. Thus, it suffices to show that $Z(\overrightarrow{K}_{2n}) = n$. In order to prove this, we let $S \subset V$ be a set of cardinality $n - 1$ of blue vertices and show it cannot be a zero forcing set.

Case 1: $S \subset L$.

By assumption, $n \geq 3$, so every blue vertex has at least two white out-neighbors in U; hence S cannot be a zero forcing set.

Case 2: $S \subset U$.

Since $|S| = n - 1$, by symmetry we assume u_0 is the only vertex in $U \setminus S$. Let $X := S \cap N^-(u_0)$ and $Y := S \setminus X$. At the first time step, the vertices in Y can force downward to color the set $Y_L \subset L$ blue; observe $\ell_{n-1} \notin Y_L$ because the only in-neighbor of ℓ_{n-1} in U is $u_0 \notin S$. At this time, no $\ell_i \in Y_L$ can force any ℓ_j since u_0 is a white out-neighbor of ℓ_i (since $i \neq n - 1$). Also, we observe that every vertex in Y has no white out-neighbors whereas every vertex in X has two white out-neighbors: One is u_0 and one is in L. Therefore the only possibility for an additional force is ℓ_i forces u_0 for some i.

Now we consider two cases and show each is impossible. First if $n = 2k + 1$ is odd, then ℓ_i has k out-neighbors in L. But Y_L, the set of blue vertices in L, contains only k vertices, including ℓ_i itself. So ℓ_i must have another white out-neighbor in L, making it impossible to force u_0. Second, if $n = 2k$ is even, then $|Y|$ can be either $k-1$ or k. If it is $k-1$, then we apply the same argument as for n odd. So we may assume $|Y| = k$, in particular, u_k is an out-neighbor of u_0 and thus $Y = \{u_1, u_2, \ldots, u_k\}$. By our construction, Y_L will be $\{\ell_0, \ell_1, \ldots, \ell_{k-1}\}$ and ℓ_k is an out-neighbor of ℓ_0. Under this assumption, for all $i \neq n - 1$, ℓ_i has at least one white out-neighbor in L, and $u_0 \in N^+(\ell_i)$. Thus for $i \neq n - 1$, ℓ_i cannot force. Since ℓ_{n-1} is white, ℓ_{n-1} cannot force. Thus S is not a zero forcing set.

Case 3: $S \cap U$ and $S \cap L$ are not empty.

We start by carrying out the following shifting process: If some $u_i \in S \cap U$ has only one white out-neighbor and it is in U, say u_j, then we replace S by $(S \setminus \{\ell_{i-1}\}) \cup \{u_j\}$. Since in the new set u_i can force ℓ_{i-1}, it is sufficient to show this new set is not a zero forcing set. Continuing this process, we may assume the set S has the property that if ℓ_{i-1} is blue, then either u_i is white or it has no unique white out-neighbor, which is in U. After completing the shifting process, assume $S \cap U$ and $S \cap L$ are not empty, or else we may apply Case 1 and Case 2. Let Y denote the set of vertices u_i in $S \cap U$ such that ℓ_{i-1} is the only white out-neighbor of u_i (it is possible that $Y = \emptyset$).

Having completed the shifting process, we claim that $|S \cap U| = n - 2$ and $|S \cap L| = 1$. To see this, first observe that $|S \cap U| \neq n - 1$, for otherwise $S \cap L = \emptyset$. Suppose $|S \cap U| \leq n - 3$. Then initially there are at least three white vertices in U; every vertex in L will have at least two white out-neighbors in U, so no vertex in L can perform a force at time $t = 1$. Also, no vertex in U can force a vertex in U since we finished the shifting process. If $Y = \emptyset$, then no forces occur, so assume $Y \neq \emptyset$. The vertices in Y are the only vertices that can perform a force at $t = 1$, and the vertices in Y will force downward to color the set $Y_L \subset L$ blue. Let $u_i \in U$ be a blue vertex. If $u_i \in Y$, then u_i has no white out-neighbor after the forces at time $t = 1$. If $u_i \notin Y$ and ℓ_{i-1} is white, then u_i has at least one white out-neighbor in $U \setminus S$. If $u_i \notin Y$ and ℓ_{i-1} is blue, then u_i has at least two white out-neighbors in $U \setminus S$, since we finished the shifting process. This means at the second step, a blue vertex in U cannot perform a force. Since each vertex in L still has two or more white out-neighbors in U, no vertex in L can perform a force either. This means the whole process stops after time $t = 1$ with some white vertices still remaining, a contradiction.

Without loss of generality, we may now assume $U \setminus S = \{u_0, u_j\}$ for some j, (u_0, u_j) is an arc, and $S \cap L = \{\ell_i\}$ for some i. If $i = j - 1$ and S is a zero forcing set, then $(S \setminus \{\ell_i\}) \cup \{u_j\}$ is also a zero forcing set. This is because $(S \setminus \{\ell_i\}) \cup \{u_j\}$ can immediately carry out the force $u_j \to \ell_i$. However, $(S \setminus \{\ell_i\}) \cup \{u_j\}$ cannot be a zero forcing set by Case 2. Next we claim that if $i \neq j - 1$, then the process stops after time $t = 1$. The only vertices that can perform forces at $t = 1$ are vertices in Y (again assume $Y \neq \emptyset$, since otherwise the process already failed), and these vertices force downward to color Y_L blue. After the first time step, every vertex in Y_L has at least two white out-neighbors, u_0 and u_j. No blue vertex in U has a unique white out-neighbor (because of the shifting process done originally and the forces done at $t = 1$). If $u_{i+1} \notin \{u_0, u_j\}$, then ℓ_i has at least two white out-neighbors so cannot force. We have already shown that $i \neq j - 1$, leaving the case $i = n - 1$. So assume $i = n - 1$. We claim ℓ_i has two white out-neighbors ℓ_0 and u_j. This is because the arc (u_0, u_j) implies either $j = 1$ or (u_1, u_j) is an arc, but both of these cases mean u_1 is not in Y (since either $u_1 = u_j$ is white, or $u_1 \neq u_j$ initially has at least two white out-neighbors, u_j and ℓ_0) and so ℓ_0 is white after $t = 1$. Therefore, the process stops after $t = 1$ with white vertices remaining. This completes Case 3.

In every case, a set S of cardinality $n - 1$ cannot be a zero forcing set. $\square$

Theorem 3.6. *For all integers $n \geq 2$, $n \neq 4, 5$, there is an orientation $\overrightarrow{K}_n$ for K_n such that* $\mathrm{pt}(\overrightarrow{K}_n) = 1$.

Proof. We have already seen that this statement is true for even n. For the case $n = 2m + 1$, we construct $\overrightarrow{K}_{2m+1}$ by adding one vertex x to an orientation of $\overrightarrow{K}_{2m}$ constructed as in Proposition 3.5, and adding directed arcs from x to all vertices in $\overrightarrow{K}_{2m}$.

Table 1
Number of connected graphs on n vertices with given the propagation time as the minimum over all orientations.

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$\min_{\vec{G}} \operatorname{pt}(\vec{G})=1$	1	2	5	20	106	820	10746	256568
$\min_{\vec{G}} \operatorname{pt}(\vec{G})=2$	0	0	1	1	6	33	371	4512

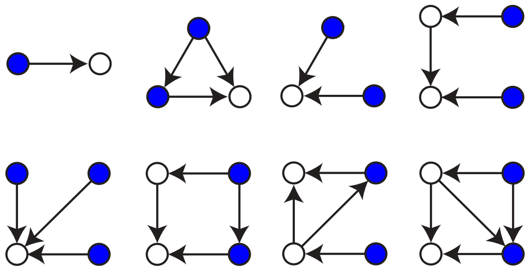

Fig. 3. The connected graphs of order at most four (other than K_4) with orientations having propagation time one.

Since the case $\vec{K}_3$ is trivial, we assume $m \geq 3$. In this case, every vertex in $\vec{K}_{2m}$ has in-degree (within $\vec{K}_{2m}$) at least one by the construction in Proposition 3.5. Let B be a minimum zero forcing set for $\vec{K}_{2m+1}$. Since $\deg^-(x) = 0, x \in B$. Since $\deg^+(x) = 2m$, x cannot perform a force until all but one vertex of $\vec{K}_{2m} = \vec{K}_{2m+1} - x$ are blue, at which point another vertex can perform the force, since the in-degree within $\vec{K}_{2m}$ is positive for every vertex in $\vec{K}_{2m}$. So $B \setminus \{x\}$ is a zero forcing set for $\vec{K}_{2m}$, implying $|B \setminus \{x\}| \geq Z(\vec{K}_{2m}) = m$ and $|B| \geq m+1$. Therefore the propagation time of $\vec{K}_{2m+1}$ is one, using the efficient zero forcing set $B'' := B' \cup \{x\}$, where B' is an efficient zero forcing set for $\vec{K}_{2m}$. $\square$

3.3. *Data for small graphs that allow propagation time one*

We say that a simple graph G *allows propagation time one* if there is some orientation $\vec{G}$ of the graph G with $\operatorname{pt}(\vec{G}) = 1$, and say G *requires propagation time at least* k if there is no orientation $\vec{G}$ of the graph G with $\operatorname{pt}(\vec{G}) \leq k-1$. Then natural questions are, "Which graphs allow propagation time one?" and, "How common are such graphs?"

We use $\min_{\vec{G}} \operatorname{pt}(\vec{G})$ to denote the minimum propagation time of $\vec{G}$ where $\vec{G}$ runs over all orientations of G. Then 'G allows propagation time one' is equivalent to $\min_{\vec{G}} \operatorname{pt}(\vec{G}) = 1$ and 'G requires at least propagation time k' is equivalent to $\min_{\vec{G}} \operatorname{pt}(\vec{G}) \geq k$. For all connected graphs of order at most nine, the minimum propagation time over all orientations was determined by use of a *Sage* program [9] run on a computer, and we present the data in Table 1. Note that no graph of order at most nine requires a propagation time of three or greater. At least for small graphs it appears that allowing propagation time one is common.

We have already established that K_4 does not allow propagation time one. A similar case analysis shows that K_5 also does not allow propagation time one. In Fig. 3 we give orientations for all the remaining connected graphs on at most 4 vertices, and have marked corresponding orientations and minimum zero forcing sets, to verify that they have $\operatorname{pt}(\vec{G}) = 1$. We remark here that the undirected graph underlying the oriented graph in Fig. 1 does not allow an orientation with propagation time one.

In addition to the data given in the table, we have verified by computer that no graph of order 10 requires propagation time of three or greater. This leads to the following open questions.

Question 3.7. *Does there exist an undirected graph G with $\min_{\vec{G}} \operatorname{pt}(\vec{G}) \geq 3$? More generally, does there exist an undirected graph G with $\min_{\vec{G}} \operatorname{pt}(\vec{G}) \geq k$ for k arbitrarily large?*

4. Orientations with high propagation times

In this section we focus on orientations of graphs that have high propagation times. The two key elements to obtain high propagation time are a small zero forcing set and few simultaneous forces occurring at each time step. Hessenberg paths play a central role. Although our primary interest is oriented graphs, much of the literature deals with simple digraphs.

Combining [14, Lemma 2.15] (which shows that $Z(\Gamma) = 1$ if and only if Γ is a Hessenberg path) and Observation 1.2 we have the following.

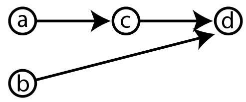

Fig. 4. A counterexample to the converse of Observation 4.4.

Observation 4.1. *For any simple digraph Γ, the following are equivalent:*

1. $Z(\Gamma) = 1$.
2. $\mathrm{pt}(\Gamma) = |\Gamma| - 1$.
3. *Γ is a Hessenberg path.*

One natural question is whether a graph G can be oriented in such a way as to produce a specific propagation time. In studying high propagation time, we ask which graphs G can be oriented to produce $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 1$. A *Hamilton path* in a graph G is a subgraph that is a path and includes all vertices of G.

Proposition 4.2. *A graph G has an orientation $\overrightarrow{G}$ with $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 1$ if and only if G has a Hamilton path.*

Proof. If $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 1$, then $\overrightarrow{G}$ is a Hessenberg path on $(v_1, v_2, \ldots, v_n)$, in which case $(v_1, v_2, \ldots, v_n)$ is a Hamilton path of G. If $(v_1, v_2, \ldots, v_n)$ is a Hamilton path of G, then we can orient G so that $\overrightarrow{G}$ is a Hessenberg path on $(v_1, v_2, \ldots, v_n)$ by choosing the arcs (v_i, v_{i+1}) for $i = 1, \ldots, n-1$, and for every edge between v_i and v_j with $j > i + 1$ choosing the back arc (v_j, v_i). Since $\overrightarrow{G}$ is a Hessenberg path, $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 1$. $\square$

A simple digraph Γ is a *digraph of two parallel Hessenberg paths* [6] if Γ is not itself a Hessenberg path and the vertices of Γ can be partitioned as $V(\Gamma) = V^{(1)} \dot\cup V^{(2)}$, with the notation $V^{(h)} = \{p_1^{(h)}, \ldots, p_{s_h}^{(h)}\}$ for $h = 1, 2$, so the following are satisfied:

(i) $P^{(h)} = \Gamma[V^{(h)}]$ is a Hessenberg path, and
(ii) there are no i, j, k, ℓ with $i < j, k < \ell$, and $(p_k^{(1)}, p_j^{(2)}), (p_i^{(2)}, p_\ell^{(1)}) \in E(\Gamma)$ (in other words, there are no forward crossing arcs between the two Hessenberg paths).

Theorem 4.3. [6] *For any simple digraph Γ, $Z(\Gamma) = 2$ if and only if Γ is a digraph of two parallel Hessenberg paths.*

If Γ is a digraph of two parallel Hessenberg paths on the Hessenberg paths $P^{(h)}$, $h = 1, 2$, we let $\Gamma(P^{(1)}, P^{(2)})$ refer to this particular way of decomposing Γ as a digraph of two parallel Hessenberg paths. In this case, $B = \{p_1^{(1)}, p_1^{(2)}\}$ is a minimum zero forcing set for Γ. Moreover, if $B' = \{v_1, v_2\}$ is any minimum zero forcing set for Γ, then Γ can be expressed as $\Gamma(P'^{(1)}, P'^{(2)})$ for two Hessenberg paths $P'^{(1)}$ and $P'^{(2)}$ with $p'^{(1)}_1 = v_1$ and $p'^{(2)}_1 = v_2$.

An *oriented graph of two parallel Hessenberg paths* $\overrightarrow{G}$ is a digraph of two parallel Hessenberg paths with no double arcs. The next statement is now immediate from Observations 1.2 and 4.1, and Theorem 4.3.

Observation 4.4. *For any oriented graph $\overrightarrow{G}$, if $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 2$, then $Z(\overrightarrow{G}) = 2$ and $\overrightarrow{G}$ is an oriented graph of two parallel Hessenberg paths.*

As is the case for graphs, the converse of Observation 4.4 is false, as shown in the next example.

Example 4.5. Let $\overrightarrow{P_4}$ be the oriented graph in Fig. 4. Then $Z(\overrightarrow{P_4}) = 2$ because vertices a and b have in-degree zero (and $\overrightarrow{P_4}$ is an oriented graph of two parallel Hessenberg paths), but $\mathrm{pt}(\overrightarrow{P_4}) = 1 \neq |\overrightarrow{P_4}| - 2$.

By Observation 4.4, to achieve $\mathrm{pt}(\overrightarrow{G}) = |\overrightarrow{G}| - 2$ it must be the case that $Z(\overrightarrow{G}) = 2$, and for every minimum zero forcing set exactly one force occurs at each time step.

We extend the notation and definitions in [15] to oriented graphs, but there are some significant differences caused by the orientation, so the definition of zig-zag path in Definition 4.6 and the conditions in Theorem 4.7 are somewhat different from those in [15]. Suppose $\overrightarrow{G}(P^{(1)}, P^{(2)})$ is an oriented graph of two parallel Hessenberg paths. The notation $x \prec y$ means x and y are on the same path $P^{(h)}$ and for some $i < j$, $x = p_i^{(h)}$ and $y = p_j^{(h)}$. Whenever the order of a subset of vertices of $P^{(h)}$ is discussed, the order is the path order. For $i > 1$, we say that $p_{i-1}^{(h)} = \mathrm{prev}(p_i^{(h)})$ and $\mathrm{next}(p_{i-1}^{(h)}) = p_i^{(h)}$. Furthermore, $\mathrm{alt}(z_i)$ denotes the out-neighbors of z_i not in the same Hessenberg path as z_i.

Definition 4.6. An orientation of two parallel Hessenberg paths $\overrightarrow{G}(P^{(1)}, P^{(2)})$ is a *zig-zag orientation*, denoted by $\overrightarrow{G}(P^{(1)}, P^{(2)}, Q)$, if $\overrightarrow{G}(P^{(1)}, P^{(2)})$ contains a directed path $Q = (z_1, z_2, \ldots, z_r)$ satisfying the following conditions:

1. $z_i \in V^{(1)}$ for i odd, $z_i \in V^{(2)}$ for i even.
2. $z_i \prec z_{i+2}$ for $1 \le i \le r-2$.
3. For $1 \le i \le r-1$, if $u \in \mathrm{alt}(z_i)$ then $u \preceq z_{i+1}$.
4. $z_1 = p_1^{(1)}$.
5. z_r is the last vertex of its path.
6. If $r = 1$, then one of
   (a) $\mathrm{alt}(z_1) = \emptyset$ or
   (b) $\mathrm{alt}(z_1) \ne \emptyset$, and for z the last vertex in $\mathrm{alt}(z_1)$, z is not the first vertex in its path and $\mathrm{prev}(z) \in \mathrm{alt}(z_1) \cup \{p_1^{(2)}\}$.
7. If $r \ge 2$, then
   (a) $z_2 \ne p_1^{(2)}$.
   (b) z_{r-1} is not the last vertex of its path.
   (c) if $\mathrm{alt}(z_r) \ne \emptyset$, then either
      i. $u \preceq z_{r-1}$ for all $u \in \mathrm{alt}(z_r)$ or
      ii. for the last vertex z in $\mathrm{alt}(z_r)$, $z \succ z_{r-1}$ and $\mathrm{prev}(z) \in \mathrm{alt}(z_r) \cup \{z_{r-1}\}$.

Subject to the constraint of being an orientation of two parallel Hessenberg paths, extra arcs that are not part of either Hessenberg path or the zig-zag path are permitted.

The following theorem characterizes whether a zero forcing set B of cardinality two achieves $\mathrm{pt}(\overrightarrow{G}, B) = |\overrightarrow{G}| - 2$, which can be attained only by having exactly one force at each step. The key ideas are: Forcing occurs only within the paths $P^{(1)}$ and $P^{(2)}$. Forcing starts with the first force on $P^{(2)}$. Forcing alternates between these two paths if $r \ge 2$; specifically, when z_i, $i \ge 2$ is forced, then the next force must occur on the other path as the current vertex on the path has out-degree at least two (coming from $P^{(i)}$ and Q). Note that the zig-zag path described in the theorem, chosen to capture the alternating forcing between paths property, is usually not the only zig-zag path for this zero forcing set.

Theorem 4.7. *Let $\overrightarrow{G}$ be a connected oriented graph of order $n \ge 3$ with $Z(\overrightarrow{G}) = 2$ and minimum zero forcing set B. Then $\mathrm{pt}(\overrightarrow{G}, B) = |\overrightarrow{G}| - 2$ if and only if $\overrightarrow{G}$ can be written as a zig-zag orientation $\overrightarrow{G}(P^{(1)}, P^{(2)}, Q)$ with $B = \{p_1^{(1)}, p_1^{(2)}\}$.*

Proof. Assume $\overrightarrow{G}$ can be written as a zig-zag orientation $\overrightarrow{G}(P^{(1)}, P^{(2)}, Q)$. The requirements of being a zig-zag orientation will require that only one force may occur at each time step (and $P^{(1)}, P^{(2)}$ can be the forcing chains). The first force occurs on $P^{(2)}$ and (if $r \ge 2$) the forcing switches between the paths immediately after a vertex z_i in Q is forced.

Conversely, if $\mathrm{pt}(\overrightarrow{G}, B) = |\overrightarrow{G}| - 2$ and $Z(\overrightarrow{G}) = 2$, the set of forces of B induce two parallel Hessenberg paths $P^{(1)}$ and $P^{(2)}$, with $B = \{p_1^{(1)}, p_1^{(2)}\}$ as the zero forcing set. All forces are of the form $p_i^{(h)} \to p_{i+1}^{(h)}$ for some i and h, and exactly one force occurs at each time step. Since $n \ge 3$, a force must occur. Without loss of generality, we assume that $p_1^{(2)} \to p_2^{(2)}$ is the first force and set $z_1 = p_1^{(1)}$.

In the case forcing occurs only in $P^{(2)}$ (i.e., $p_1^{(1)}$ is the only vertex in $P^{(1)}$), set $r = 1$. This means $p_1^{(1)} = z_1$ has no out-neighbors (implying $\mathrm{alt}(z_1) = \emptyset$), or it is impossible for $p_1^{(1)}$ to perform a force until possibly the last step (in which case we have chosen to have a vertex in $P^{(2)}$ perform the last force rather than having $p_1^{(1)}$ perform the last force). This implies 6(b) from the definition of a zig-zag orientation (note that $\mathrm{alt}(z_1) = \{p_1^{(2)}\}$ is not permitted, because this would cause $Z(\overrightarrow{G}) = 1$).

Now assume forcing takes place in both paths. We identify the remaining vertices z_i in Q by using the propagation process. At any point in the propagation process, only one of the paths is forcing, i.e., is active, and the other cannot force, i.e., is inactive. The path $P^{(2)}$ is initially active and forcing will continue along $P^{(2)}$ until some vertex, which we label z_2, is reached and the path $P^{(2)}$ becomes inactive, and $P^{(1)}$ becomes active. We continue this process of identifying the zig-zag path by choosing as z_i the vertex that was forced immediately before the path that is active switches. This process of labeling continues until the last switch between active and inactive paths occurs; the last vertex where a switch happens is labeled z_r; observe $r \ge 2$.

Since z_i could not force until z_{i+1} turned blue, the directed arc (z_i, z_{i+1}) must be present; this shows that these arcs form a directed path Q. By definition $z_1 = p_1^{(1)}$, while the labeling will have $z_2 = p_k^{(2)}$ for some $k > 1$, the z_i alternates between the two paths, $z_i \prec z_{i+2}$, and z_r is the final vertex on one of the paths while z_{r-1} is not the final vertex on its path (because it performs a force immediately after z_r turns blue). To ensure $\overrightarrow{G}(P^{(1)}, P^{(2)}, Q)$ is a zig-zag orientation, we need to verify condition (3) of Definition 4.6. Suppose $z_{i+1} \prec u$ for some $u \in \mathrm{alt}(z_i)$ with $1 \le i \le r-1$, then at the step right after z_{i+1} turns blue, z_i has at least two white out-neighbors, namely $\mathrm{next}(z_i)$ and u. If this happens, then the forcing process will stop here, since we picked z_{i+1} as the vertex that cannot conduct the force $z_{i+1} \to \mathrm{next}(z_{i+1})$ right after $\mathrm{prev}(z_{i+1}) \to z_{i+1}$. This is a contradiction, so condition (3) of Definition 4.6 holds, and we have our zig-zag orientation.

We now need to verify that this zig-zag orientation satisfies the properties given in Definition 4.6. By construction (6) does not apply, so it remains to show that all the parts of (7) hold. Properties 7(a) and 7(b) are satisfied because a force must take place in $P^{(2)}$ before z_2 is defined, and all remaining forces occur on the other path after z_r is defined. For 7(c), assume $\mathrm{alt}(z_r) \ne \emptyset$ and the last vertex z in $\mathrm{alt}(z_r)$ is after z_{r-1}. Finally we have $\mathrm{prev}(z) = z_{r-1}$ or $\mathrm{prev}(z) \in \mathrm{alt}(z_r)$. □

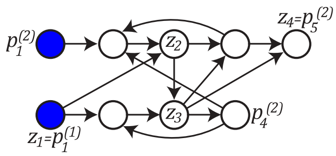

Fig. 5. An oriented graph satisfying the properties of Theorem 4.7.

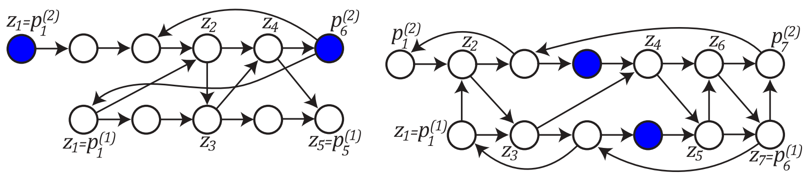

Fig. 6. Examples of zig-zag orientations conforming to Theorem 4.7 and showing other minimum zero forcing sets that have propagation time less than $|\overrightarrow{G}|-2$.

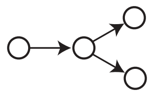

Fig. 7. An oriented graph with two minimum zero forcing sets and propagation time $|\overrightarrow{G}|-2$.

Corollary 4.8. *Let $\overrightarrow{G}$ be an orientation of a connected graph G for which $Z(\overrightarrow{G})=2$. Then $\mathrm{pt}(\overrightarrow{G})=|\overrightarrow{G}|-2$ if and only if for every minimum zero forcing set B, $\overrightarrow{G}$ can be written as a zig-zag orientation $\overrightarrow{G}(P^{(1)},P^{(2)},Q)$.*

The oriented graph $\overrightarrow{G}$ shown in Fig. 5 illustrates Corollary 4.8. The set $B=\{p_1^{(1)},p_1^{(2)}\}$ is a zero forcing set for $\overrightarrow{G}$ that satisfies the hypotheses of Theorem 4.7, and B is the only minimum zero forcing set (since $p_1^{(1)}$ and $p_1^{(2)}$ both have in-degree zero). Thus $\mathrm{pt}(\overrightarrow{G})=|\overrightarrow{G}|-2$.

In order to guarantee $\mathrm{pt}(\overrightarrow{G})=|\overrightarrow{G}|-2$, it is not enough to be able to write $\overrightarrow{G}$ as a zig-zag orientation $\overrightarrow{G}(P^{(1)},P^{(2)},Q)$ satisfying the properties of Theorem 4.7 for some $B=\{p_1^{(1)},p_2^{(1)}\}$. Although $\mathrm{pt}(\overrightarrow{G},B)=|\overrightarrow{G}|-2$, there may be another minimum zero forcing set B' for which $\mathrm{pt}(\overrightarrow{G},B')<|\overrightarrow{G}|-2$; Fig. 6 presents several such examples.

Note that these examples have the property that one or both of the initial vertices of the $P^{(i)}$ have positive in-degree. In the case when neither of the initial vertices have any in-arcs, then B is the unique minimum zero forcing set and so it is enough that there is a zig-zag orientation starting with B that satisfies the hypotheses of Theorem 4.7. However, having a unique minimum zero forcing set is not necessary for an oriented graph to have $\mathrm{pt}(\overrightarrow{G},B)=|\overrightarrow{G}|-2$: The digraph $\overrightarrow{K}_{1,3}$ shown in Fig. 7 has two minimum zero forcing sets and $\mathrm{pt}(\overrightarrow{K}_{1,3})=2=4-2$. The problem of giving a complete classification of all oriented graphs with $\mathrm{pt}(\overrightarrow{G})=|\overrightarrow{G}|-2$ remains open.

5. Orientation propagation intervals

We have seen in preceding sections that for the path P_n there are orientations with both high and low propagation times. This leads to the following idea.

Definition 5.1. Let G be an undirected graph with $m=\min_{\overrightarrow{G}}\mathrm{pt}(\overrightarrow{G})$ and $M=\max_{\overrightarrow{G}}\mathrm{pt}(\overrightarrow{G})$. The interval $[m,M]$ is called the *orientation propagation interval*, and G has a *full* orientation propagation interval if for every k such that $m\le k\le M$ there is some orientation $\overrightarrow{G}$ such that $\mathrm{pt}(\overrightarrow{G})=k$.

Determining if a graph has a full orientation propagation interval is non-trivial, even for some simple graphs. The difficulty is that the propagation parameter can be sensitive to small perturbations, as shown in the following example.

Example 5.2. Let $n\ge 9$ and $k=\lfloor\frac{n+3}{2}\rfloor$. Consider the two oriented paths on n vertices shown in Fig. 8 where the vertices are labeled 1 to n going from left to right. The top path has $Z(\overrightarrow{P}_n)=2$, $\{1,k\}$ is the unique minimum zero forcing set, and $\overrightarrow{P}_n$ has propagation time $n-2$. The bottom path has $Z(\overrightarrow{P}_n)=3$ and $\{1,k,k+1\}$ is a minimum zero forcing set with propagation time of $\lceil\frac{n-5}{2}\rceil$. Thus the reversal of the arc between $k-2$ and $k-1$ changed the propagation time by at least $\lfloor\frac{n+1}{2}\rfloor$, which can be arbitrarily large.

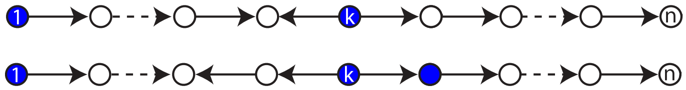

Fig. 8. Reversing an arc produces a large change in propagation time.

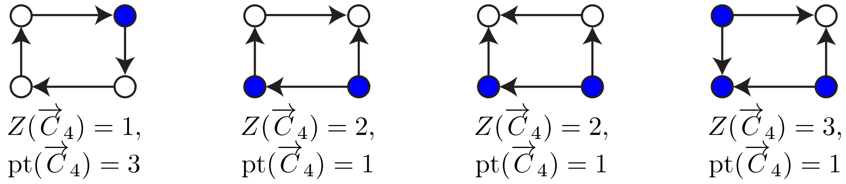

Fig. 9. Possible orientations of C_4.

In this section we will show that paths have full orientation propagation time interval, while cycles do not. We comment that the behavior and analysis of orientation propagation time intervals are far from understood.

5.1. Paths have a full orientation propagation time interval

By Theorem 3.3 we know for P_n that there is an orientation with propagation time one, and if we orient the edges of the path as $(i, i+1)$ for $1 \le i \le n-1$, then the propagation time is $n-1$. We will show that for P_n there is an orientation with propagation time k for each $1 \le k \le n-1$. The remaining propagation times are given in the proof of the following theorem.

Theorem 5.3. *Let P_n be the path on n vertices and $1 \le k \le n-1$. Then there is an orientation $\overrightarrow{P}_n$ such that $\mathrm{pt}(\overrightarrow{P}_n) = k$.*

Proof. We label the path with vertices $1, \ldots, n$ and edges joining i and $i+1$ for $1 \le i \le n-1$. To achieve a propagation time of $n-2$ for $n \ge 4$, take the orientation $(i, i+1)$ for $1 \le i \le n-3$ together with $(n-1, n-2)$ and $(n-1, n)$. Then this orientation $\overrightarrow{P}_n$ has $Z(\overrightarrow{P}_n) = 2$, and $\{1, n-1\}$ is the unique minimum zero forcing set; furthermore, no simultaneous forces can occur, giving propagation time $n-2$.

Now assume $2 \le k \le n-3$ and $n \ge 5$. We consider the following orientation.

- $(i, i+1)$ for $1 \le i \le k+1$ (the initial segment).
- For $k+2 \le j \le n-1$ orient the edge between j and $j+1$ by

$$\begin{cases} (j, j+1) & \text{if } j \equiv k \text{ or } k+1 \pmod 4, \\ (j+1, j) & \text{if } j \equiv k+2 \text{ or } k+3 \pmod 4. \end{cases}$$

A minimum zero forcing set must contain 1, but no other vertex in the initial segment (as 1 can eventually force the initial segment). In particular, vertex $k+1$ will not turn blue until the kth step of the propagation process (the vertex $k+2$ can be turned blue earlier through its other neighbor). Therefore the propagation time of this orientation is at least k.

Consider the set $S = \{i : \deg^-(i) = 0\}$. The vertices in S must be in a zero forcing set since they cannot be turned blue by a neighbor. If a vertex in S has $\deg^+(i) = 2$ then one of the neighbors must also be in the zero forcing set, i.e., only i can change them to blue, but it cannot force both. As a consequence when we look at blocks of consecutive vertices between vertices with $\deg^-(i) = 2$ we see that each block will have two elements in the zero forcing set and the block will propagate in time two. Similar analysis shows that the tail will also propagate in time at most two.

We conclude that in two steps all vertices except some of those in the initial segment of the path have been turned blue; however, the initial segment will not finish turning blue until time k, so the propagation time of this orientation of P_n is k. $\square$

5.2. Cycles do not have a full orientation propagation time interval

Not all graphs have a full orientation propagation interval, as the following example shows.

Example 5.4. The four orientations on C_4 up to isomorphism are shown in Fig. 9. No orientation has a propagation time of 2, although there are orientations with propagation times 1 and 3. So C_4 does not have a full orientation propagation interval.

If we orient the cycle C_n by $(i, i+1)$ for $1 \le i \le n$ (where we look at the entries modulo n), then any vertex can force the entire graph and has $\mathrm{pt}(\overrightarrow{C}_n) = n-1$.

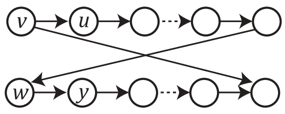

Fig. 10. A cycle oriented as two parallel Hessenberg paths.

Now we reverse the arc between 1 and n to $(1, n)$. Use the zero forcing set $B = \{1, 2\}$. Vertices 3 and n are forced in the first step. Then at each subsequent step one vertex is forced, so B has propagation time $n - 3$. Every minimum zero forcing set for this orientation is of the form $\{1, k\}$, and for $k \geq 3$ this set has propagation time $n - 2$. Thus $\mathrm{pt}(\overrightarrow{C}_n) = n - 3$.

However, the intermediate value of $n - 2$ is impossible, as the next result shows. In particular, for $n \geq 4$ the cycle C_n does not have a full orientation propagation time interval.

Proposition 5.5. *Let $n \geq 4$. Then $\mathrm{pt}(\overrightarrow{C}_n) \neq n - 2$ for any orientation of C_n.*

Proof. Suppose $\overrightarrow{C}_n$ is an orientation of C_n with $\mathrm{pt}(\overrightarrow{C}_n) = n - 2$. By Observation 1.2 we must have that $Z(\overrightarrow{C}_n) = 2$. Moreover, it must be the case that precisely one vertex is forced at each time step for every minimum zero forcing set. Since $Z(\overrightarrow{G}) = 2$, $\overrightarrow{G}$ is a graph of two parallel Hessenberg paths, with two arcs between the two paths so that a cycle is formed. Since the initial vertices v and w of the two Hessenberg paths must be a zero forcing set and cannot both force initially, without loss of generality the arcs must be oriented as shown in Fig. 10.

First suppose an out-neighbor u of v exists in the path containing v. Then $\{v, u\}$ is a zero forcing set in which two forces are performed initially, contradicting $\mathrm{pt}(\overrightarrow{G}) = n - 2$. Thus u does not exist and $\{v, w\}$ is a zero forcing set in which two forces are performed initially (since $n \geq 4$, y is not the last vertex in the lower path), contradicting $\mathrm{pt}(\overrightarrow{G}) = n - 2$. $\square$

6. Throttling on oriented graphs

To this point we have focused on the propagation time for zero forcing sets that have minimum cardinality. We can relax the requirement to use minimum zero forcing sets and more generally consider any set that forces the entire graph. In this situation there are several possible questions that could be investigated. Here we want to minimize the sum of the cardinality of the zero forcing set and the speed at which it propagates through the graph, following the study of throttling for undirected graphs in [10]. This study began in response to a question asked by Richard Brualdi during a presentation about propagation time by Michael Young at the 2011 ILAS conference in Braunschweig, Germany. One perspective on this question is to assume a cost to initially color a vertex and also a cost in waiting for all vertices to become blue; obviously other weights could also be considered. A related question is discussed in Section 7.

Definition 6.1. Given an oriented graph $\overrightarrow{G}$ and a zero forcing set B of $\overrightarrow{G}$, the *throttling time* of B for $\overrightarrow{G}$ is $\mathrm{th}(\overrightarrow{G}, B) = \mathrm{pt}(\overrightarrow{G}, B) + |B|$. The *minimum throttling time* of an oriented graph $\overrightarrow{G}$ is

$$\mathrm{th}(\overrightarrow{G}) = \min\{\mathrm{pt}(\overrightarrow{G}, B) + |B| : B \text{ is a zero forcing set of } \overrightarrow{G}\}.$$

In [10] the (undirected version of the) throttling time of an undirected path P_n was determined to be approximately $2\sqrt{n}$. More generally it was shown that for any fixed value k there is a constant c_k such that if the zero forcing number of a graph on n vertices is at most k, then the minimum throttling time is at most $c_k\sqrt{n}$. These results are best possible, up to a constant, since the minimum possible throttling time of any n-vertex graph is $2\sqrt{n} - 1$. On the other hand it is easy to come up with graphs with throttling time which is linear in the number of vertices, e.g., large random graphs have this property since a linear number of vertices are required to even carry out one force.

Here we look at throttling on complete Hessenberg paths and show that unlike undirected graphs, we cannot guarantee a significant savings in throttling. A *complete Hessenberg path* is the unique tournament with a zero forcing number of one. More precisely, the complete Hessenberg path of order n has vertex set $\{1, 2, \ldots, n\}$ and the following arcs:

$$\{(i, i + 1) : 1 \leq i \leq n - 1\} \cup \{(i, j) : 3 \leq i \leq n \text{ and } 1 \leq j \leq i - 2\}.$$

For $n \geq 4$, a simple check verifies that $\{1\}$ is the unique minimum zero forcing set of this oriented graph. We show an example with $n = 5$ in Fig. 11.

In particular, for the complete Hessenberg path $\overrightarrow{\mathcal{H}}$, we show that $\mathrm{th}(\overrightarrow{\mathcal{H}}) = \lfloor \frac{2n}{3} \rfloor + 1$. The key step is given in the next lemma, which shows that we cannot engage in a large number of simultaneous forces on the complete Hessenberg path.

Lemma 6.2. *Let $\overrightarrow{\mathcal{H}}$ be a complete Hessenberg path, and B a set of blue vertices. Then B can force at most 2 vertices at any given time step.*

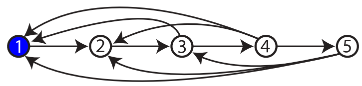

Fig. 11. A complete Hessenberg path on 5 vertices.

Proof. Let $\overrightarrow{\mathcal{H}}$ be a complete Hessenberg path with vertex set $\{1, 2, \ldots, n\}$ and assume B is a set of blue vertices that forces 3 or more vertices at time step t. Assume $a < b < c$ are the largest of the vertices that are forced at time t and thus are white at time $t - 1$. Observe that c can only be forced by vertex $c - 1$ or by some vertex $c + 2$ or greater; but any vertex $c + 2$ or greater has a, b and c as white out-neighbors so cannot perform any forces. This means $c - 1$ must force c. Since a vertex must be blue to force, $a < b < c - 1$. Thus $c - 1$ has both a and c as white out-neighbors so cannot perform any forces. This means that no vertex can force vertex c, which is a contradiction, and the result follows. □

Corollary 6.3. *For any zero forcing set B of the complete Hessenberg path $\overrightarrow{\mathcal{H}}$, we have* $2\,\mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + |B| \geq n$.

In the remainder of this section we will find it convenient for the proofs to group the vertices of the complete Hessenberg path on n vertices into sets of three. We will adopt the following notation: $\ell := \lfloor n/3 \rfloor$ and for $1 \leq j \leq \ell$ then $I_j = \{3j - 2, 3j - 1, 3j\}$ while $I_{\ell+1}$ will be the remaining vertices (if any).

Remark 6.4. Note that any zero forcing set must contain at least one vertex in I_1: if this were not the case, then we could never change I_1 to blue because every vertex not in I_1 is adjacent to two elements in I_1.

Proposition 6.5. *If $\overrightarrow{\mathcal{H}}$ is a complete Hessenberg path with $|\overrightarrow{\mathcal{H}}| = n$ then* $\mathrm{th}(\overrightarrow{\mathcal{H}}) \leq \left\lfloor \frac{2n}{3} \right\rfloor + 1$.

Proof. Define $B := \{1, 3, 6, 9, \ldots, 3\ell\}$ and note that $|B| = \left\lfloor \frac{n}{3} \right\rfloor + 1$. An easy inductive argument shows that vertices $3t - 2$ and $3t$ will force vertices $3t - 1$ and $3t + 1$ in tth propagation round until no more forcing can occur, with the exception that when $n \equiv 0, 2 \bmod 3$ there will be a single force in the last step. For $n \equiv 1 \bmod 3$, the $\frac{2(n-1)}{3}$ white vertices are forced in $\frac{n-1}{3} = \left\lfloor \frac{n}{3} \right\rfloor$ time steps. For $n \equiv 0 \bmod 3$, the $\frac{2n}{3} - 1$ white vertices are forced in $\frac{n}{3} = \left\lfloor \frac{n}{3} \right\rfloor$ time steps, with the last step forcing only one vertex. Thus,

$$\mathrm{th}(\overrightarrow{\mathcal{H}}, B) = \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right) + \left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

For $n \equiv 2 \bmod 3$, the $2\left\lfloor \frac{n}{3} \right\rfloor + 1$ white vertices are forced in $\left\lfloor \frac{n}{3} \right\rfloor + 1$ time steps, so $\mathrm{pt}(\overrightarrow{\mathcal{H}}, B) = \left\lfloor \frac{n}{3} \right\rfloor + 1$ and

$$\mathrm{th}(\overrightarrow{\mathcal{H}}, B) = \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right) + \left(\left\lfloor \frac{n}{3} \right\rfloor + 1 \right) = \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

In all cases $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) = \left\lfloor \frac{2n}{3} \right\rfloor + 1$. □

Lemma 6.6. *If $\overrightarrow{\mathcal{H}}$ is a complete Hessenberg path on n vertices and B is a zero forcing set such that $|B| \leq \lfloor \frac{n}{3} \rfloor$ then there is some time step when exactly one vertex is forced.*

Proof. By Remark 6.4, $I_1 \cap B \neq \emptyset$. Suppose first that for some $2 \leq m \leq \ell$ that $I_m \cap B = \emptyset$. Now we set $S = I_1 \cup \cdots \cup I_{m-1}$ and $T = I_m \cup \cdots \cup I_{\ell+1}$. Since every blue vertex in T is adjacent to two white vertices in I_m, then nothing in T can force until $3m - 2$ (the first element of T) has been forced blue. Furthermore, when $3m - 2$ has been forced, necessarily everything in S has also already been forced: in order for $3m - 3$ to force, $1, 2, \ldots, 3m - 5$ must be blue; if $3m - 4$ is blue before $3m - 3$ forces then the result holds, and otherwise $3m - 5$ will force $3m - 4$ at the same time that $3m - 3$ forces $3m - 2$. At this stage $3m - 2$ can force $3m - 1$ and the only other vertex that can possibly be adjacent to only one white vertex is $3m + 1$ adjacent to $3m - 1$. Therefore, the only vertex that is forced at this stage is $3m - 1$.

If all $I_m \cap B \neq \emptyset$ then it must be that $|I_m \cap B| = 1$ for $1 \leq m \leq \ell$. Therefore every vertex numbered 4 or greater has at least two white out-neighbors and so only the single blue vertex in I_1 can force at the first step. □

Theorem 6.7. *If $\overrightarrow{\mathcal{H}}$ is a complete Hessenberg path then* $\mathrm{th}(\overrightarrow{\mathcal{H}}) = \left\lfloor \frac{2n}{3} \right\rfloor + 1$.

Proof. An easy verification establishes the result when the complete Hessenberg path $\overrightarrow{\mathcal{H}}$ has 1, 2 or 3 vertices. We will now proceed by induction on n, the number of vertices of $\overrightarrow{\mathcal{H}}$. Assume that $\mathrm{th}(\overrightarrow{\mathcal{H}}) = \lfloor 2k/3 \rfloor + 1$ for $1 \leq k \leq n - 1$. Consider the complete Hessenberg path on n vertices and for the sake of contradiction we will assume we can find a zero forcing set B of $\overrightarrow{\mathcal{H}}$ with $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \leq \left\lfloor \frac{2n}{3} \right\rfloor$. Then by Corollary 6.3 and this assumption we have

$$n \leq \mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + \mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + |B| \leq \mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + \left\lfloor \frac{2n}{3} \right\rfloor \leq \mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + \frac{2n}{3}.$$

This tells us that $\frac{n}{3} \leq \mathrm{pt}(\overrightarrow{\mathcal{H}}, B)$, which in turn implies that $|B| \leq \lfloor \frac{n}{3} \rfloor$. We now consider two cases (which exhaust all possibilities by Remark 6.4).

Case 1: $I_m \cap B = \emptyset$ for some $2 \leq m \leq \ell$.

Proceeding as in the previous lemma we let $S = I_1 \cup \cdots \cup I_{m-1}$ and $T = I_m \cup \cdots \cup I_{\ell+1}$ and note that no force happens from the blue vertices in T until all vertices in S and $x := 3m - 2$ (the first vertex in T) have been forced. Once x has been forced the rest of the forcing can proceed as though S is not part of the graph. This shows we can split the propagation into two distinct phases and so we have

$$\mathrm{pt}(\overrightarrow{\mathcal{H}}, B) = \mathrm{pt}(\overrightarrow{\mathcal{H}}[S \cup \{x\}], B \cap S) + \mathrm{pt}(\overrightarrow{\mathcal{H}}[T], (B \cap T) \cup \{x\}).$$

Using this we get the following bound:

$$\begin{aligned}
\mathrm{th}(\overrightarrow{\mathcal{H}}, B) &= \mathrm{pt}(\overrightarrow{\mathcal{H}}, B) + |B| \\
&= \mathrm{pt}(\overrightarrow{\mathcal{H}}[S \cup \{x\}], B \cap S) + \mathrm{pt}(\overrightarrow{\mathcal{H}}[T], (B \cap T) \cup \{x\}) + |B \cap S| + |B \cap T| \\
&= \mathrm{th}(\overrightarrow{\mathcal{H}}[S \cup \{x\}], B \cap S) + \mathrm{th}(\overrightarrow{\mathcal{H}}[T], (B \cap T) \cup \{x\}) - 1 \\
&\geq \lfloor 2(|S| + 1)/3 \rfloor + 1 + \lfloor 2(n - |S|)/3 \rfloor + 1 - 1 \\
&\geq \left\lfloor \frac{2n}{3} \right\rfloor + 1.
\end{aligned}$$

The -1 term on the third line comes from accounting for $\{x\}$, and on the fourth line we have used the induction hypothesis. This statement contradicts that $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \leq \lfloor \frac{2n}{3} \rfloor$, so this case cannot happen.

Case 2: $I_m \cap B \neq \emptyset$ for all $1 \leq m \leq \ell$.

In this case we must then have that $|B| = \lfloor \frac{n}{3} \rfloor$ and so

$$\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \geq 1 + \frac{n - \lfloor \frac{n}{3} \rfloor - 1}{2} + \left\lfloor \frac{n}{3} \right\rfloor.$$

The first two terms are a lower bound for $\mathrm{pt}(\overrightarrow{\mathcal{H}}, B)$, since by Lemma 6.6 there is some time step in the forcing process when only one force occurs, and since for every other time step at most two forces can occur by Lemma 6.2. Since $\mathrm{th}(\overrightarrow{\mathcal{H}}, B)$ must be a whole number this then implies that $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \geq \lfloor \frac{2n}{3} \rfloor + 1$, which is again a contradiction to $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \leq \lfloor \frac{2n}{3} \rfloor$.

Therefore we can conclude that $\mathrm{th}(\overrightarrow{\mathcal{H}}, B) \geq \left\lfloor \frac{2n}{3} \right\rfloor + 1$ and the construction in Proposition 6.5 is tight. □

7. Discussion

The emphasis in this paper has been on choosing an orientation of a graph to give a particular propagation time, and in Section 5 we examined the possible propagation times over various orientations. Other perspectives include determining the propagation time for a given oriented graph or family of graphs, and considering the propagation times of various minimum zero forcing sets for a given oriented graph or family of graphs. Here we discuss these ideas briefly.

To aid in the computation of propagation time of a specific oriented graph, we have published a *Sage* program that allows computation of the zero forcing number and propagation time of a simple digraph [9].

Computation of propagation time of certain families of graphs is done in [11]. A (labeled) graph with m edges has 2^m orientations, many of which are non-isomorphic. Thus the problem of determining propagation times over various orientations of a family of graphs for any but the most obvious very sparse families such as paths and cycles, discussed in Section 5, is a rather large problem.

Given a fixed graph, questions such as the various propagation times of its minimum zero forcing sets, techniques for minimizing or maximizing propagation time among minimum zero forcing sets, and whether there are gaps in the interval of propagation times of minimum zero forcing sets, are studied in [15]. The analog for oriented graphs involves fixing the orientation as well as the graph. As with graphs, it is easy to find an example of an oriented graph that has minimum zero forcing sets with different propagation times. For example, the second orientation of C_4 shown in Fig. 9 has another minimum zero forcing set with propagation time 2, whereas the minimum zero forcing set shown has propagation time 1.

One (theoretical) technique to find a minimum zero forcing set with minimum propagation time that works for both graphs and oriented graphs is to search for (or design) a zero forcing set that allows multiple simultaneous forces. Of course in practice this involves carrying out the propagation process, so this is not of great practical use, although it does play a role in designing graphs or orientations that achieve low propagation times, or high propagation times by avoiding multiple forces.

Regarding the issue of gaps in the interval of possible propagation times of minimum zero forcing sets for a fixed digraph, the existence of such an example for graphs (Example 1.10 in [15]) immediately yields an example of a simple digraph with such a gap, by doubly directing the graph (replacing each edge $\{u, v\}$ by the two arcs (u, v) and (v, u)). The next example shows that such gaps also exist for oriented graphs.

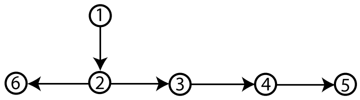

Fig. 12. An oriented tree with a gap in propagation times of minimum zero forcing sets.

Example 7.1. Consider the oriented tree $\overrightarrow{T}$ shown in Fig. 12. The only minimum zero forcing sets of $\overrightarrow{T}$ are $B_1 = \{1, 3\}$ and $B_2 = \{1, 6\}$. The propagation times are $\mathrm{pt}(\overrightarrow{T}, B_1) = 2$ and $\mathrm{pt}(\overrightarrow{T}, B_2) = 4$.

Another generalization of throttling to consider is to determine the minimum propagation time over all zero forcing sets of size k (for k at least as large as the zero forcing number). As an example, consider the complete Hessenberg path of length n. There is a unique zero forcing set of size 1, which has a propagation time $n-1$. From Lemma 6.2 we note that at most two forces can occur at a time, and moreover will only occur if the vertices $1, 2, \ldots, \ell$ and $\ell+2$ are blue and $\ell+1$ and $\ell+3$ is white. In particular, for $2 \le k \le \left\lfloor \frac{n-1}{3} \right\rfloor + 1$ we initially color the vertices $1, 3, 6, \ldots, 3k-3$ blue allowing us to double force in $k-1$ rounds and then single force for the remainder giving minimum propagation time $n-2k+1$. For $\left\lfloor \frac{n-1}{3} \right\rfloor + 2 \le k \le n-1$ then the optimal strategy is to set up as many double forces as possible (following the above strategy with the tail colored blue), and in particular has propagation time $\left\lfloor \frac{n-k+1}{2} \right\rfloor$. Finally if all vertices are colored blue, then propagation time is 0. To summarize, the minimum propagation time of a zero forcing set of size k for the complete Hessenberg path is as follows:

$$\begin{cases} n-2k+1 & \text{if } 1 \le k \le \left\lfloor \dfrac{n-1}{3} \right\rfloor + 1, \\ \left\lfloor \dfrac{n-k+1}{2} \right\rfloor & \text{if } \left\lfloor \dfrac{n-1}{3} \right\rfloor + 2 \le k \le n-1, \\ 0 & \text{if } k = n. \end{cases}$$

It might be interesting to see what similar analysis gives for other classes of graphs (including for undirected graphs).

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