arXiv:1604.08817v1 [math.CO] 29 Apr 2016

Multi-part Nordhaus-Gaddum type problems for tree-width, Colin de Verdière type parameters, and Hadwiger number

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May 2, 2016

Abstract

A traditional Nordhaus-Gaddum problem for a graph parameter β is to find a (tight) upper or lower bound on the sum or product of $\beta(G)$ and $\beta(\overline{G})$ (where $\overline{G}$ denotes the complement of G). An r-decomposition $G_1, \dots, G_r$ of the complete graph K_n is a partition of the edges of K_n among r spanning subgraphs $G_1, \dots, G_r$. A traditional Nordhaus-Gaddum problem can be viewed as the special case for $r = 2$ of a more general r-part sum or product Nordhaus-Gaddum type problem. We determine the values of the r-part sum and product upper bounds asymptotically as n goes to infinity for the parameters tree-width and its variants largeur d'arborescence, path-width, and proper path-width. We also establish ranges for the lower bounds for these parameters, and ranges for the upper and lower bounds of the r-part Nordhaus-Gaddum type problems for the parameters Hadwiger number, the Colin de Verdière number μ that is used to characterize planarity, and its variants ν and ξ.

Keywords. Nordhaus-Gaddum, multi-part, Hadwiger number, tree-width, largeur d'arborescence, path-width, proper path-width, Colin de Verdière type parameter.

AMS subject classifications. 05C35, 05C83, 05C69, 05C50

1 Introduction

An r-*decomposition* of $K_n = ([n], E)$ is a partition of the edges as the edge sets of r spanning subgraphs $G_i = ([n], E_i)$ for $i = 1, \dots, r$. An r-part Nordhaus-Gaddum problem for a graph parameter β is to find a (tight) upper or lower bound on the sum

$$\beta(G_1) + \cdots + \beta(G_r)$$

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or on the product

$$\beta(G_1)\cdots\beta(G_r).$$

It is common to exclude from these bounds graphs of order $\leq n_0$ for a fixed n_0, or to ask that the bound be approached for arbitrarily large n.

The study of Nordhaus-Gaddum type problems for more than two parts was initiated by Füredi et al. in [10], and we follow the definitions of that paper. Let β be a nonnegative integer valued graph parameter, and let r and n be positive integers. Define the Nordhaus-Gaddum sum upper bound and sum lower bound:

$$\beta(r;n) := \max\{\beta(G_1) + \cdots + \beta(G_r)\}$$

where the maximum is taken over all r-decompositions of K_n;

$$\underline{\beta}(r;n) := \min\{\beta(G_1) + \cdots + \beta(G_r)\}$$

where the minimum is taken over all r-decompositions of K_n (in [10] this lower value $\underline{\beta}$ is denoted by $\overline{\beta}$, but we believe a lower line is more mnemonic for a lower value).

Define the Nordhaus-Gaddum product upper bound and product lower bound:

$$\beta^{\times}(r;n) = \max\{\beta(G_1)\cdots\beta(G_r)\}$$

where the maximum is taken over all r-decompositions of K_n;

$$\underline{\beta}^{\times}(r;n) = \min\{\beta(G_1)\cdots\beta(G_r)\}$$

where the minimum is taken over all r-decompositions of K_n. For the product lower bound, because many of the parameters take on the value zero on an edgeless graph, we focus on the non-degenerate bound $\underline{\beta}^{\times}_{\mathrm{nd}}(r;n)$, in which every graph in a decomposition must have an edge (see Section 1.2).

Since $\beta(1;n) = \beta(K_n)$ and similarly for the other bounds defined, we study decompositions with $r \geq 2$. There is a rich literature on 2-part Nordhaus-Gaddum type problems (see [1] for a survey). We study r-part Nordhaus-Gaddum type problems for the following parameters: the Hadwiger number η, tree-width tw and its variants largeur d'arborescence la, path-width pw, and proper path-width ppw, and the Colin de Verdière type parameters μ, ν, ξ, all of which are minor monotone and have interesting relationships (see [3] for a discussion of these relations). Definitions of the parameters are given in Section 1.1. Kostochka says [15, p. 307], "it is very important to study the Hadwiger number" due to its relation to other ideas in graph theory, including Hadwiger's famous conjecture. He establishes extensive Nordhaus-Gaddum theory ($r = 2$) for η in [14, 15, 16]. Likewise, tree-width and its variants have played a fundamental role in the theory of graph minors since their (re)introduction by Robertson and Seymour in the early 1980s. Surprisingly, Nordhaus-Gaddum theory ($r = 2$) has only recently been studied for tree-width and its variants, with the sum lower bound established in [7, 13] and the sum upper bound established in [13]. In addition to other uses in graph theory that motivated their introduction, Colin de Verdière type parameters have played an important role in the study of minimum rank/maximum nullity of real symmetric matrices described by a graph (see [3, 8, 12]). The Nordhaus-Gaddum sum lower bound ($r = 2$) is called the Graph Complement Conjecture in minimum rank literature (see [8]).

Here we state our main results. We begin with the sum upper bound.

Theorem 1.1.

(a) For a fixed $r \geq 2$ *and* β *one of tree-width* tw *or its variants largeur d'arborescence* la*, path-width* pw*, or proper path-width* ppw*,*

$$\lim_{n\to\infty} \frac{\beta(r;n)}{n} = r.$$

(b) For a fixed $r \geq 2$ *and* β *one of the Hadwiger number* η *or the Colin de Verdière type parameters* μ, ν, ξ*,* $\beta(r; n) = \Theta(n)$*. Specifically, for* $n \geq 3$ *as* $n \to \infty$

$$\frac{r}{\left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil} - o(1) \leq \frac{\beta(r;n)}{n} \leq \sqrt{r} + o(1).$$

Theorem 1.1 is established by Theorems 2.1, 2.2, 2.4, and Corollary 2.5. It should be noted that Theorem 1.1(a) was established for $r = 2$ and tree-width in [13] and better results are known for Theorem 1.1(b) in the case that $r = 2$: Kostochka [14] showed that for $n \geq 5$, $\eta(2; n) = \left\lfloor \frac{6}{5}n \right\rfloor$, so $\lim_{n\to\infty} \frac{\eta(2;n)}{n} = \frac{6}{5} > 1 = \frac{2}{\left\lceil \sqrt{2\cdot 2+\frac{1}{4}} - \frac{1}{2} \right\rceil}$. Barrett et al. [4] showed that $\frac{4}{3} \leq \limsup_{n\to\infty} \frac{\beta(2;n)}{n} \leq \sqrt{2}$ for $\beta = \xi$ or ν. The fact that $\lim_{n\to\infty} \frac{\eta(2;n)}{n} < \limsup_{n\to\infty} \frac{\beta(2;n)}{n}$ for $\beta \in \{\xi, \nu\}$ suggests that the upper bound in Theorem 1.1(b) is not tight for η and the lower bound is not tight for $\beta \in \{\xi, \nu\}$, since our upper bound for η is obtained from our upper bound for ξ and our lower bound for ν and ξ is obtained from our lower bound for η.

Next we consider the sum lower bound.

Theorem 1.2. *For a fixed* $r \geq 2$ *and* $n \to \infty$*,* $\underline{\eta}(r; n) = \Theta(\frac{n}{\sqrt{\log n}})$*. Specifically, for* n *large enough,*

$$\frac{1}{570r}\frac{n}{\sqrt{\log n}} \leq \underline{\eta}(r;n) \leq r\frac{n}{\sqrt{\log n}}.$$

Theorem 1.2 is established by Theorem 3.1 using results of Kostochka [15].

It is known [7, 12, 13] that $\underline{\beta}(2; n) = n - 2$ for $\beta \in \{\text{tw}, \text{la}, \text{pw}, \text{ppw}\}$, and conjectured that $\underline{\beta}(2; n) = n - 2$ for $\beta \in \{\mu, \nu, \xi\}$ (see [8, 12] and the references therein). This does not generalize to $r > 2$.

Theorem 1.3. *For* $\beta \in \{\text{tw}, \text{la}, \text{pw}, \text{ppw}\}$ *and* $r \geq 3$*,* $\underline{\beta}(r; n) = \Theta(n)$*. Specifically, as* $n \to \infty$*,*

$$\frac{1}{2} - o(1) < r - \sqrt{r^2 - r} - o(1) \leq \frac{\underline{\beta}(r;n)}{n} \leq \frac{3}{4} + o(1).$$

Theorem 1.3 is established by Corollaries 3.3 and 3.5. Since $\eta(G) - 1 \leq \nu(G) \leq \xi(G) \leq \text{ppw}(G)$ and $\eta(G) - 1 \leq \mu(G) \leq \xi(G) \leq \text{ppw}(G)$ for any graph that has an edge, Theorems 1.2 and 1.3 imply lower and upper bounds on for $\beta \in \{\mu, \nu, \xi\}$.

Next we turn our attention to product bounds, beginning with the upper bound and followed by the non-degenerate lower bound.

Theorem 1.4.

(a) For a fixed $r \geq 2$ and β one of tree-width tw *or its variants* la, pw, *or* ppw,

$$\lim_{n\to\infty} \frac{\beta^{\times}(r;n)}{n^r} = 1.$$

(b) For a fixed $r \geq 2$ and β one of the Hadwiger number η or the Colin de Verdière type parameters ν, μ, ξ, $\beta^{\times}(r;n) = \Theta(n^r)$. Specifically, as $n \to \infty$,

$$\left\lceil \sqrt{2r + \frac{1}{4}} - \frac{1}{2} \right\rceil^{-r} - o(1) \leq \frac{\beta^{\times}(r;n)}{n^r} \leq (\sqrt{r})^{-r} + o(1).$$

Theorem 1.4 is established by Theorems 4.2 and 4.3. Again, better results are known for Theorem 1.1(b) in the case that $r = 2$: Kostochka [14] showed that for $n \geq 5$, $\eta(2;n) = \left\lfloor \frac{1}{4} \left\lfloor \frac{6n}{5} \right\rfloor^2 \right\rfloor$, so $\lim_{n\to\infty} \frac{\eta^{\times}(2;n)}{n^2} = \frac{9}{25} > \frac{1}{4} = \left\lceil \sqrt{2 \cdot 2 + \frac{1}{4}} - \frac{1}{2} \right\rceil^{-2}$. Barrett et al. [4] showed that $\frac{4}{9} \leq \limsup_{n\to\infty} \frac{\beta(2;n)}{n} \leq \frac{1}{2}$ for $\beta = \xi$ or ν. Again the upper bound in Theorem 1.1(b) is likely not tight for η and the lower bound is likely not tight for $\beta \in \{\xi, \nu\}$.

Theorem 1.5. *For a fixed $r \geq 2$ and β one of the Hadwiger number η, the Colin de Verdière type parameters ν, μ, ξ, tree-width* tw *or its variants* la, pw, *or* ppw, $\underline{\beta}^{\times}_{\mathrm{nd}}(r;n) = \Theta(n)$. *Specifically:*

(a) Let $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$. For $n \geq 4$, $\underline{\beta}^{\times}_{\mathrm{nd}}(2;n) = n - 3$ [12]. For $r \geq 3$ as $n \to \infty$,

$$\frac{1}{2} - o(1) \leq \frac{\underline{\beta}^{\times}_{\mathrm{nd}}(r;n)}{n} \leq 1,$$

(b) For $\beta \in \{\nu, \xi, \mu\}$, $r \geq 2$, and $n \geq 1$,

$$\frac{1}{2^{2r-2}} \leq \frac{\underline{\beta}^{\times}_{\mathrm{nd}}(r;n)}{n} \leq 1.$$

(c) For $r \geq 2$ and $n \geq 1$,

$$(0.513)^{r-2} \leq \frac{\underline{\eta}^{\times}_{\mathrm{nd}}(r;n)}{n} \leq 2^{r-1}.$$

Theorem 1.5 is established by Theorem 5.2 and Corollaries 5.6 and 5.7. In the case of the Hadwiger number, the general lower bound, which permits graphs with no edges, is also of interest.

Theorem 1.6. *For a fixed $r \geq 2$, $\underline{\eta}^{\times}(r;n) = \Theta(n)$. Specifically,*

$$(0.513)^{r-2} \leq \frac{\underline{\eta}^{\times}(r;n)}{n} \leq 1.$$

Theorem 1.6 is established by Theorem 5.4, with the case $\underline{\eta}^{\times}(2;n) = n$ following from results of Kostochka [16] (see Remark 5.3). We also make a conjecture about the product lower bound for η, which is implied by Hadwiger's Conjecture (see Remark 5.5).

Conjecture 1.7. *For all $r \geq 2$ and $n \geq 1$, $\underline{\eta}^{\times}(r;n) = n$.*

1.1 Definitions of parameters and notation

All graphs are simple, undirected, and finite, G denotes a graph, and n denotes the order of G. A more complete description of the parameters discussed here (and the graph notation we use) can be found in [3] and [12]. The *clique number* $\omega(G)$ is the maximum order of a clique in G and the *Hadwiger number* $\eta(G)$ is the maximum order of a clique minor of G.

A graph consisting of isolated vertices has tree-width zero (and similarly for the variants of tree-width defined here), so in the remainder of this paragraph we assume a graph has an edge. Let k be a positive integer. A *k-tree* is constructed inductively by starting with a complete graph on $k+1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices, so a tree of order at least two is a 1-tree. The *tree-width* $\mathrm{tw}(G)$ is the minimum k for which G is a subgraph of a k-tree. Every k-tree has at least two vertices of degree k. The maximal cliques of a k-tree are of order $k+1$, and the *facets* of a maximal clique are its k-clique subgraphs. A *linear k-tree* is constructed inductively by starting with K_{k+1} and connecting each new vertex to a facet that includes the vertex added in the previous step. The *proper path-width* $\mathrm{ppw}(G)$ is the minimum k for which G is a subgraph of a linear k-tree. A *k-caterpillar* is constructed by starting with K_{k+1} and at each stage adding a new maximal clique by adjoining a new vertex to the k vertices of some facet of the maximal clique that was added in the previous step. The *path-width* $\mathrm{pw}(G)$ is the minimum k for which G is a subgraph of a k-caterpillar. A *two-sided k-tree* is constructed by starting with K_{k+1} and connecting each new vertex to the vertices of an existing K_k that either includes a vertex of degree k or is the same as the K_k to which some previous vertex was connected. The *largeur d'arborescence* $\mathrm{la}(G)$ is the minimum k for which G is a subgraph of a two-sided k-tree. Clearly $\mathrm{tw}(G) \leq \mathrm{la}(G) \leq \mathrm{pw}(G) \leq \mathrm{ppw}(G)$ for every graph G. It is known that $\mathrm{la}(G) \leq \mathrm{tw}(G) + 1$ [5] and $\mathrm{ppw}(G) \leq \mathrm{pw}(G) + 1$ [18] for every graph G. A more comprehensive discussion of these parameters, including justification for some of the equivalent definitions used here, is given in [3].

The Colin de Verdière type parameters are linear algebraic graph parameters. All matrices discussed are real and symmetric; the set of $n \times n$ real symmetric matrices is denoted by $S_n(\mathbb{R})$. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph* of A is $\mathcal{G}(A) = ([n], E)$ where $E = \{\{i,j\} : i, j \in [n], i \neq j, \text{ and } a_{ij} \neq 0\}$; the diagonal of A is ignored in determining $\mathcal{G}(A)$. The *set of symmetric matrices described by* G is $\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}$. A real symmetric matrix A satisfies the *Strong Arnold Property* provided there does not exist a nonzero real symmetric matrix X satisfying $AX = O$, $A \circ X = O$, and $I \circ X = O$, where $\circ$ denotes the entry-wise product, i.e., $(A \circ B)_{ij} = a_{ij} b_{ij}$, I is the identity matrix, and O is the zero matrix. The parameter $\xi(G)$ is the maximum nullity among matrices $A \in \mathcal{S}(G)$ satisfying the Strong Arnold Property. The parameter $\nu(G)$ is the maximum nullity among positive semidefinite matrices $A \in \mathcal{S}(G)$ satisfying the Strong Arnold Property (a matrix $A \in S_n(\mathbb{R})$ is *positive semidefinite* if all eigenvalues of A are nonnegative). The Colin de Verdière number $\mu(G)$ is defined to be the maximum nullity among symmetric matrices $A = [a_{ij}] \in \mathcal{S}(G)$ such that A satisfies the Strong Arnold Hypothesis, A has exactly one negative eigenvalue, and A is a generalized Laplacian (i.e., for all $i \neq j$, $a_{ij} \leq 0$).

Asymptotic comparisons arise naturally in our work, and since some of the notation has more than one interpretation in the papers cited, we state our notation here. Let f and g be real valued functions of $\mathbb{N}$. We say: f is $o(g)$ if $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| = 0$; f is $O(g)$ if there exist constants C, N such that $|f(n)| \le C|g(n)|$ for all $n \ge N$; f is $\Omega(g)$ if there exist constants C, N such that $|f(n)| \ge C|g(n)|$ for all $n \ge N$; f is $\Theta(g)$ if f is $O(g)$ and $\Omega(g)$.

1.2 Decompositions and non-degeneracy

A *random r-decomposition* is an r-decomposition of K_n in which each G_i is a $G(n, \frac{1}{r})$ random graph (with the understanding that necessarily the G_i are dependent).

Lemma 1.8. *For fixed r and n large, a random r-decomposition exists.*

Proof. Each edge of K_n is assigned by an independent r-valued variable that determines which of the graphs $G_i, i = 1, \dots, r$, is assigned the edge. Thus $G_i, i = 1, \dots, r$, is an r-decomposition of K_n and each G_i has probability $\frac{1}{r}$ for each edge, i.e., G_i is $G(n, \frac{1}{r})$. □

Some of the results we use require a graph to have an edge. A *non-degenerate r-decomposition* of $K_n = ([n], E)$ is a partition of the edges as the edge sets of r spanning subgraphs $G_i = ([n], E_i), i = 1, \dots, r$ with $E_i \ne \emptyset$ for $i = 1, \dots, r$. We define non-degenerate versions of the four bounds, $\beta_{\text{nd}}(r; n), \underline{\beta}_{\text{nd}}(r; n), \beta^{\times}_{\text{nd}}(r; n)$, and $\underline{\beta}^{\times}_{\text{nd}}(r; n)$, where the maximum or minimum is taken over all non-degenerate r-decompositions of K_n.

Suppose $G_1, G_2, \dots, G_r$ is an r-decomposition on n vertices. If in this decomposition there are exactly ℓ graphs each having at least one edge, we may assume that $G_{\ell+1} = \dots = G_r = \overline{K_n}$ are empty graphs, so that $G_1, \dots, G_\ell$ form a non-degenerate ℓ-decomposition.

Observation 1.9. Assuming $\beta(\overline{K_n}) = 0$ or $\beta(\overline{K_n}) = 1$, the relationship between degenerate and non-degenerate sum bounds is:

$$\beta(r; n) = \begin{cases} \max\limits_{1\le\ell\le n}\{\beta_{\text{nd}}(\ell; n) + (r-\ell)\beta(\overline{K_n})\} & \text{if } \beta(\overline{K_n}) = 1, \\ \max_{1\le\ell\le n}\{\beta_{\text{nd}}(\ell; n)\} & \text{if } \beta(\overline{K_n}) = 0; \end{cases}$$

$$\underline{\beta}(r; n) = \begin{cases} \min\limits_{1\le\ell\le n}\{\underline{\beta}_{\text{nd}}(\ell; n) + (r-\ell)\beta(\overline{K_n})\} & \text{if } \beta(\overline{K_n}) = 1, \\ \min_{1\le\ell\le n}\{\underline{\beta}_{\text{nd}}(\ell; n)\} & \text{if } \beta(\overline{K_n}) = 0; \end{cases}$$

Observation 1.10. Assuming $\beta(\overline{K_n}) = 0$ or $\beta(\overline{K_n}) = 1$, the relationship between degenerate and non-degenerate product bounds is:

$$\beta^{\times}(r; n) = \begin{cases} \max\limits_{1\le\ell\le r}\{\beta^{\times}_{\text{nd}}(\ell; n)\} & \text{if } \beta(\overline{K_n}) = 1, \\ \beta^{\times}_{\text{nd}}(r; n) & \text{if } \beta(\overline{K_n}) = 0; \end{cases}$$

$$\underline{\beta}^{\times}(r; n) = \begin{cases} \min\limits_{1\le\ell\le r}\{\underline{\beta}^{\times}_{\text{nd}}(\ell; n)\} & \text{if } \beta(\overline{K_n}) = 1, \\ 0 & \text{if } \beta(\overline{K_n}) = 0. \end{cases}$$

In the case of the product lower bound for parameters having $\beta(\overline{K_n}) = 0$, the non-degenerate parameter is clearly the more interesting one.

2 Nordhaus-Gaddum Sum Upper Bounds

For all the parameters $\beta \in \{\eta, \mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, $\beta(G) \leq n$, so $\beta(r;n) \leq rn$. For tree-width and its variants largeur d'arborescence, path-width and proper path-width, this is essentially the best we can do.

Theorem 2.1. *For a fixed* $r \geq 2$, $\mathrm{tw}(r;n) = rn - o(n) = \mathrm{la}(r;n) = \mathrm{pw}(r;n) = \mathrm{ppw}(r;n)$.

Proof. For tree-width, this follows from the fact that for fixed $p > 0$, $\mathrm{tw}(G(n,p)) = n - o(n)$ [17], the existence of a random r-decomposition (Lemma 1.8), and the linearity of expectation (see, for example, [20, Lemma 8.5.7]). The remaining statements follow from the fact that $\mathrm{ppw}(G) \geq \mathrm{pw}(G) \geq \mathrm{la}(G) \geq \mathrm{tw}(G)$ for all G. □

Next we consider the Colin de Verdière type parameters and Hadwiger number.

Theorem 2.2. *Let* $\beta \in \{\xi, \nu, \mu\}$. *For a fixed* $r \geq 2$ *and* $n \geq 2\sqrt{r}$, $\beta(r;n) \leq \sqrt{r}n$. *For a fixed* $r \geq 2$ *and* $n \geq 2\sqrt{r}$, $\eta(r;n) \leq \sqrt{r}\,n + r$.

Proof. We begin by establishing the statement for ξ. It is shown in [11] that for any graph $G = (V(G), E(G))$, $|E(G)| \geq \frac{\xi(G)(\xi(G)+1)}{2} - 1$. In [4] it is noted that this implies $\xi(G)^2 \leq 2|E(G)|$ for every graph G that has an edge. Let $G_1, \ldots, G_r$ be a non-degenerate r-decomposition of K_n. Let $\mathbb{1}$ denote the all ones vector of length r and $\vec{\xi} := [\xi(G_1), \ldots, \xi(G_r)]^T$. Then

$$\|\vec{\xi}\|_2^2 = \xi(G_1)^2 + \cdots + \xi(G_r)^2 \leq 2(|E(G_1)| + \cdots + |E(G_r)|) \leq 2\binom{n^2}{2} = n^2.$$

That is, $\|\vec{\xi}\|_2 \leq n$. By the Cauchy-Schwarz inequality,

$$\xi(G_1) + \cdots + \xi(G_r) = \mathbb{1}^T\vec{\xi} \leq \|\mathbb{1}\|_2\|\vec{\xi}\|_2 \leq \sqrt{r}\,n,$$

so $\xi_{\mathrm{nd}}(r;n) \leq \sqrt{r}\,n$.

Now assume $n \geq 2\sqrt{r}$. By Observation 1.9,

$$\xi(r;n) = \max_{1\leq \ell \leq n}\{\xi_{\mathrm{nd}}(\ell;n) + (r-\ell)\xi(\overline{K_n})\} \leq \max_{1\leq \ell\leq n}\{\sqrt{\ell}\,n + r - \ell\} \leq \sqrt{r}\,n.$$

The last inequality follows from the assumption that $n \geq 2\sqrt{r}$ by elementary algebra.

Since $\nu(G) \leq \xi(G)$ and $\mu(G) \leq \xi(G)$ for all graphs G, $\nu(r;n) \leq \xi(r;n)$ and $\mu(r;n) \leq \xi(r;n)$ for all $r \geq 2$ and n. Since $\eta(G) - 1 \leq \nu(G)$ for all graphs G, $\eta(r;n) \leq \nu(r;n) + r$ for all $r \geq 2$ and n. □

Remark 2.3. For any minor monotone parameter β, $n \geq m$ implies $\beta(r;n) \geq \beta(r;m)$, because we can take a decomposition $G_1, \ldots, G_r$ that realizes $\beta(r;m)$, add the extra vertices to every G_i, and allocate the extra edges among the G_i however we choose without lowering $\beta(G_1) + \cdots + \beta(G_r)$.

A *triangular number* is a number of the form $\frac{t(t+1)}{2}$ for some positive integer t. For an arbitrary positive integer r, define the *triangular root* of r by $\operatorname{trt}(r) := \sqrt{2r+\frac{1}{4}} - \frac{1}{2}$, so $t = \lceil \operatorname{trt}(r) \rceil$ is equivalent to $\frac{t(t+1)}{2} < r \leq \frac{(t+1)(t+2)}{2}$.

Theorem 2.4.

(i) Fix $r \geq 2$ and define $t := \lceil \operatorname{trt}(r) \rceil$. Then for every $s \geq 1$ and $n := ts$,

$$\frac{r}{t}n + (r - t) \leq \eta(r; n).$$

(ii) For a fixed $r \geq 2$ and $n \to \infty$,

$$\frac{r}{\left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil} n - o(n) \leq \eta(r; n).$$

Proof. (i): Let $n = ts$. Partition the vertices into sets $V_i, i = 1, \ldots, t$ of s vertices each. For $i = 1, \ldots, t$ define H_i to be the subgraph of K_n induced by V_i (isomorphic to K_s) together with the $n - s$ vertices not in V_i (as isolated vertices). For $i = 1, \ldots, t-1, j = i+1, \ldots, t$ define $H_{i,j}$ to be the subgraph of K_n consisting of all n vertices and the edges having one end-point in V_i and the other in V_j; $H_{i,j}$ is isomorphic to $K_{s,s}$ and isolated vertices. This defines $\frac{t(t+1)}{2}$ subgraphs. By the choice of t, $r \leq \frac{t(t+1)}{2}$. This allows us to assign $G_i = H_i$ for $i = 1, \ldots, t$ and then pick $r - t$ distinct $H_{j,k}$'s as $\{G_i\}_{i=t+1}^{r}$. Put every edge not in any of $\{G_i\}_{i=1}^{t}$ in G_1 so that $\{G_i\}_{i=1}^{t}$ forms an r-decomposition. Now $\eta(G_i) \geq \eta(K_s) = s$ for $i = 1, \ldots, t$ and $\eta(G_i) \geq \eta(K_{s,s}) = s + 1$ for $s = t+1, \ldots, r$ (with the latter done by contracting a matching of cardinality $s - 1$). Therefore,

$$\eta(r; n) \geq \sum_{i=1}^{r} \eta(G_i) \geq ts + (r - t)(s + 1) = rs + (r - t) \geq \frac{r}{t}n + (r - t).$$

(ii): Define $t := \lceil \operatorname{trt}(r) \rceil$, and for $n \geq t$, define $q := \left\lfloor \frac{n}{t} \right\rfloor$ and $m := qt$. Note that $n - m < t$. By Remark 2.3 and part (i),

$$\eta(r; n) \geq \eta(r; m) \geq \frac{r}{t}m = \frac{r}{t}n - \frac{r}{t}(n - m) \geq \frac{r}{t}n - r = \frac{r}{t}n - o(n). \qquad \square$$

Since $\eta(G) - 1 \leq \mu(G) \leq \xi(G)$, $\eta(r; n) - r \leq \mu(r; n) \leq \xi(r; n)$ (and similarly for ν), the next corollary is immediate.

Corollary 2.5. *Let $\beta \in \{\mu, \nu, \xi\}$.*

(i) Fix $r \geq 2$ and define $t := \lceil \operatorname{trt}(r) \rceil$. Then for every $s \geq 1$ and $n := ts$,

$$\frac{r}{t}n - t \leq \beta(r; n).$$

(ii) For a fixed $r \geq 2$ and $n \to \infty$,

$$\frac{r}{\left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil} n - o(n) \leq \beta(r; n).$$

Thus, $\frac{r}{\left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil} \leq \limsup_{n\to\infty} \frac{\beta(r;n)}{n} \leq \sqrt{r}$ for $\beta \in \{\xi, \nu, \mu, \eta\}$. In Table 1 we provide the values of these lower and upper bounds for small r.

Table 1: Lower and upper bounds for $\limsup_{n\to\infty} \frac{\beta(r;n)}{n}$ when $\beta \in \{\xi, \nu, \mu, \eta\}$

r	$\frac{r}{\left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil}$	$\sqrt{r}$
3	1.5	1.73205
4	1.33333	2.
5	1.66667	2.23607
6	2.	2.44949
7	1.75	2.64575
8	2.	2.82843
9	2.25	3.
10	2.5	3.16228

3 Nordhaus-Gaddum Sum Lower Bounds

We use the ideas in Kostochka's proof that $\underline{\eta}(2;n) = \Theta(\frac{n}{\sqrt{\log n}})$ [15, Corollary 5] to establish an analogous result for $\underline{\eta}(r;n)$ with $r \geq 3$.

Theorem 3.1. *For a fixed $r \geq 2$ and $n \to \infty$, $\underline{\eta}(r;n) = \Theta(\frac{n}{\sqrt{\log n}})$. Specifically, for n large enough,*

$$\frac{1}{570r} \frac{n}{\sqrt{\log n}} \leq \underline{\eta}(r;n) \leq r \frac{n}{\sqrt{\log n}}.$$

Proof. Let $G_i = (V_i, E_i)$ be any r-decomposition of K_n. Then there must be some G_ℓ that has $|E_\ell| \geq \frac{n(n-1)}{2r}$. So by [15, Theorem 1] with $k = \frac{n-1}{2r}$,

$$\eta(G_1) + \cdots + \eta(G_r) \geq \eta(G_\ell) \geq \frac{k}{270\sqrt{\log k}} \geq \frac{n-1}{540r\sqrt{\log (n-1)}}.$$

Thus for $n \geq 19$, $\underline{\eta}(r;n) \geq \frac{1}{570r} \frac{n}{\sqrt{\log n}}$.

Almost all graphs G of order n have $\eta(G) \leq \frac{n}{\sqrt{\log n}}$ [15, p. 308]. Thus there exists an r-decomposition such that

$$\eta(G_1) + \cdots + \eta(G_r) \leq r \frac{n}{\sqrt{\log n}},$$

and $\underline{\eta}(r;n) \leq r \frac{n}{\sqrt{\log n}}$. □

The next two results establish upper bounds for the sum lower bound $\underline{\beta}(r;n)$ for $r \geq 3$ and $\beta \in \{\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, beginning with path-width.

Theorem 3.2. *For $r \geq 3$, $\underline{\mathrm{pw}}(r;n) \leq 3 \left\lceil \frac{n}{4} \right\rceil$.*

Proof. Consider the case $n = 4p$ and $r = 3$ first, and define the following 3-decomposition (illustrated in Figure 1): Partition the vertices of K_n into four sets $V_i, i = 1, \dots, 4$ of p vertices each. The edges of G_1 are the edges within V_1, the edges within V_4, the edges between V_1 and V_2, and the edges between V_3 and V_4. The edges of G_2 are the edges within V_2, the edges within V_3, the edges between V_1 and V_3, and the edges between V_2 and V_4. The edges of G_3 are the edges between V_1 and V_4, and the edges between V_2 and V_3.

G_1 G_2 G_3

Figure 1: Schematic diagram of a decomposition of K_{4p} with $\mathrm{pw}(G_1)+\mathrm{pw}(G_2)+\mathrm{pw}(G_3) = 3p$, where the shaded regions have all edges and the unshaded regions have no edges.

Observe that G_1 and G_2 each consist of two copies of the same graph H, so $\mathrm{pw}(G_1) = \mathrm{pw}(G_2) = \mathrm{pw}(H)$. Since H can be constructed by starting with a K_p and adding p additional vertices each adjacent to all the vertices of the original K_p, $\mathrm{pw}(H) = p$. Note that G_3 consists of two copies of $K_{p,p}$ and $\mathrm{pw}(K_{p,p}) = p$. Therefore,

$$\frac{3}{4}n = 3p = \mathrm{pw}(G_1) + \mathrm{pw}(G_1) + \mathrm{pw}(G_1) \geq \underline{\mathrm{pw}}(3; n).$$

For the case in which $n = 4p+\ell$ with $1 \leq \ell \leq 3$, assign $p+1$ vertices to $V_i, i = 1, \dots, \ell$ and p to the other sets. Then $\mathrm{pw}(G_i) \leq p+1 = \left\lceil \frac{n}{4} \right\rceil$, and $3\left\lceil \frac{n}{4} \right\rceil \geq \mathrm{pw}(G_1) + \mathrm{pw}(G_2) + \mathrm{pw}(G_3) \geq \underline{\mathrm{pw}}(3; n)$. For the case of $r > 3$, a degenerate decomposition can be used. □

Corollary 3.3.

(i) Let $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}\}$. *For* $r \geq 3$,

$$\underline{\beta}(r; n) \leq 3\left\lceil \frac{n}{4} \right\rceil \qquad \textit{and} \qquad \underline{\beta}_{\mathrm{nd}}(r; n) \leq 3\left\lceil \frac{n}{4} \right\rceil + r - 3.$$

(ii) For $r \geq 3$,

$$\underline{\mathrm{ppw}}(r; n) \leq 3\left\lceil \frac{n}{4} \right\rceil + r \qquad \textit{and} \qquad \underline{\mathrm{ppw}}_{\mathrm{nd}}(r; n) \leq 3\left\lceil \frac{n}{4} \right\rceil + 2r - 3.$$

Proof. The first statement in (i) follows from Theorem 3.2 together with the inequality $\mathrm{tw}(G) \leq \mathrm{la}(G) \leq \mathrm{pw}(G)$ for all graphs G. The second statement in (i) follows from the first statement by first constructing a degenerate r-decomposition based on the non-degenerate 3-decomposition in Theorem 3.2, and then removing $r-3$ edges from G_3 and placing one of these edges in each $G_i, i = 4, \dots, n$. Statement (ii) follows from statement (i) and the fact that $\mathrm{ppw}(G) \leq \mathrm{pw}(G) + 1$ for any graph G. □

Theorem 3.4. *For* $r \geq 2$ *and* $n \geq 1$,

$$\underline{\mathrm{tw}}(r; n) \geq rn - \frac{1}{2}r - \sqrt{(r^2 - r)n^2 - (r^2 - r)n + \frac{1}{4}r^2}.$$

Proof. First note that every k-tree on n vertices (necessarily $n \geq k+1$) has

$$E_k := \frac{k(k-1)}{2} + (n-k)k = -\frac{1}{2}k^2 + (n - \frac{1}{2})k$$

edges. So if $|V(G)| = n$ and $\text{tw}(G) = k$, then $|E(G)| \leq E_k$. Suppose K_n is decomposed as $\{G_i\}_{i=1}^r$, with a_k graphs having $\text{tw}(G_i) = k$ for $k = 0, \ldots, n-1$. Since every edge of K_n must be covered,

$$\sum_{k=0}^{n-1} a_k E_k \geq \frac{n(n-1)}{2}. \tag{1}$$

So we are trying to minimize $\sum_{i=1}^r \text{tw}(G_i)$ subject to inequality (1).

We begin by defining S_1 and S_2 by

$$\begin{aligned} S_1 &:= \sum_{k=0}^{n-1} a_k k = \sum_{i=1}^{r} \text{tw}(G_i); \\ S_2 &:= \sum_{k=0}^{n-1} a_k k^2. \end{aligned}$$

Since S_2 can be viewed as the sum of $r = \sum_{k=0}^{n-1} a_k$ squares, by Cauchy-Schwarz inequality,

$$\left(\sum_{k=0}^{n-1} a_k k^2\right)\left(r \cdot 1^2\right) \geq \left(\sum_{k=0}^{n-1} a_k k\right)^2.$$

This means $S_2 \geq \frac{S_1^2}{r}$.

$$\begin{aligned} \sum_{k=0}^{n-1} a_k E_k &= -\frac{1}{2}S_2 + (n - \frac{1}{2})S_1 \\ &\leq -\frac{1}{2r}S_1^2 + (n - \frac{1}{2})S_1. \end{aligned}$$

So inequality (1) implies

$$-\frac{1}{2r}S_1^2 + (n - \frac{1}{2})S_1 - \frac{n(n-1)}{2} \geq 0.$$

By solving the inequality,

$$\sum_{i=1}^{r} \text{tw}(G_i) = S_1 \geq rn - \frac{1}{2}r - \sqrt{(r^2 - r)n^2 - (r^2 - r)n + \frac{1}{4}r^2}.$$

So $\underline{\text{tw}}(r; n)$ is also greater than or equal to this value. ☐

Corollary 3.5. *Let $\beta \in \{\text{tw}, \text{la}, \text{pw}, \text{ppw}\}$. For $r \geq 2$,*

$$\liminf_{n \to \infty} \frac{\underline{\beta}(r; n)}{n} \geq r - \sqrt{r^2 - r} > \frac{1}{2}.$$

Proof. For a fixed r,

$$\lim_{n\to\infty} \frac{rn - \frac{1}{2}r - \sqrt{(r^2-r)n^2 - (r^2-r)n + \frac{1}{4}r^2}}{n} = r - \sqrt{r^2 - r} \geq \frac{1}{2},$$

with the last inequality verified by simple algebra. □

Corollary 3.6.

(i) For $r \geq 3$ and $n \geq 19$,

$$\frac{1}{570r}\frac{n}{\sqrt{\log n}} - r \leq \underline{\nu}(r;n) \leq \underline{\nu}_{\rm nd}(r;n) \leq 3\left\lceil \frac{n}{4} \right\rceil + r - 3.$$

(ii) Let $\beta \in \{\mu, \xi\}$. For $r \geq 3$ and $n \geq 19$,

$$\frac{1}{570r}\frac{n}{\sqrt{\log n}} - r \leq \underline{\beta}(r;n) \leq \underline{\beta}_{\rm nd}(r;n) \leq 3\left\lceil \frac{n}{4} \right\rceil + 2r - 3.$$

Proof. The first inequality in each statement follows from Theorem 3.1 and the inequalities $\eta(G) - 1 \leq \nu(G)$ and $\eta(G) - 1 \leq \mu(G) \leq \xi(G)$ for all graphs G. The third inequality in statement (i) (respectively, (ii)) follows from the non-degenerate case in Corollary 3.3(i) (respectively, 3.3(ii)) and the fact that for a graph G that has an edge, $\nu(G) \leq \mathrm{la}(G)$ [5] (respectively, $\mu(G) \leq \xi(G) \leq \mathrm{ppw}(G)$ [3]). □

Corollaries 3.3 and 3.6 provide our best upper bounds on $\underline{\beta}(r;n)$ as $n \to \infty$ for $r \geq 3$ and $\beta \in \{\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$. However, for n in the range $2r \leq n < 4r$ (and in some cases somewhat larger), the bound in the next theorem is better, and it is also used in Section 5.

Theorem 3.7. *Let $\beta \in \{\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$. For $r \geq 2$ and $n \geq 2r$,*

$$\underline{\beta}(r;n) \leq \underline{\beta}_{\rm nd}(r;n) \leq n - r,$$

and $n - r$ is realized by a non-degenerate decomposition in which all but at most one of the graphs are paths.

Proof. Consider proper path-width first. Define K_{2r} to be the subgraph of K_n induced by the vertices $[2r]$. We can partition the $r(2r-1)$ edges of K_{2r} into r paths each having $2r-1$ edges [19]. Without loss of generality, one of the paths, which we call the "last path," is $(1, 2, \ldots, 2r)$. Define the following non-degenerate r-decomposition of $K_n = ([n], E)$: The $r - 1$ paths of K_{2r} that are not the last path, each taken together with $n - 2r$ isolated vertices, denoted by $P_i = ([n], E_i), i = 1, \ldots, r-1$, and $P_r = \left([n], E \setminus \left(\dot\cup_{i=1}^{r-1} E_i\right)\right)$; observe that P_r includes the last path. For every $i = 1, \ldots, r-1$, $\mathrm{ppw}(P_i) = 1$. We construct P_r as a linear $(n - 2r + 1)$-tree as follows: Begin with the $(n - 2r + 2)$-clique induced by $\{2r - 1, \ldots, n\}$. For $i = 1, \ldots, 2r - 2$, add vertex $2r - 1 - i$ adjacent to $2r - i$ and to each vertex in $\{2r + 1, \ldots, n\}$. Thus $\mathrm{ppw}(P_r) = n - 2r + 1$. Since each P_i has an edge, $\underline{\mathrm{ppw}}_{\rm nd}(r,n) \leq (r-1) \cdot 1 + (n - 2r + 1) = n - r$.

Finally, $\mathrm{tw}(G) \leq \mathrm{la}(G) \leq \mathrm{pw}(G) \leq \mathrm{ppw}(G)$, $\nu(G) \leq \mathrm{la}(G)$, and $\mu(G) \leq \xi(G) \leq \mathrm{ppw}(G)$ for every graph G that has an edge, and the decomposition just constructed is non-degenerate. Thus we have $\underline{\beta}_{\rm nd}(r;n) \leq n - r$ for $\beta \in \{\mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}\}$. □

4 Nordhaus-Gaddum Product Upper Bounds

To see the relation between sums and products, we use the inequality of arithmetic and geometric means (AM-GM inequality)

$$\prod_{i=1}^{r} a_i \le r^{-r}\left(\sum_{i=1}^{r} a_i\right)^r$$

for nonnegative real numbers $a_1, a_2, \ldots, a_r$.

To establish a lower bound for a product, we use a technical lemma.

Lemma 4.1. *For fixed integers $r \ge 2$ and $n \ge 1$, let $a_1, a_2, \ldots, a_r$ be integers such that $1 \le a_i \le n$ for all i. Let $\sigma = \sum_{i=1}^{r} a_i$ and $\pi = \prod_{i=1}^{r} a_i$. Then $n^q\rho \le \pi$, where $q = \left\lfloor \frac{\sigma-r}{n-1} \right\rfloor$ and $\rho = \sigma - r - q(n-1) + 1$.*

Proof. Suppose $1 < a_1 \le a_2 < n$. Observe that

$$(a_1 - 1) + (a_2 + 1) + \sum_{i=3}^{r} a_i = \sigma$$

but

$$(a_1 - 1)(a_2 + 1)\prod_{i=1}^{r} a_i < \pi.$$

Therefore, subject to a fixed σ, the value of π is minimized when all values of a_i's but at most one are either 1 or n. This is equivalent to solving for nonnegative integers q and ρ in

$$\sigma = (r - 1 - q) \cdot 1 + qn + \rho,$$

where $1 \le \rho < n$. This equation can be rewritten as $\sigma - r = q(n-1) + (\rho - 1)$ with $0 \le \rho - 1 < n - 1$. Therefore, $q = \left\lfloor \frac{\sigma-r}{n-1} \right\rfloor$ is uniquely determined by the division algorithm, and so is $\rho = \sigma - r - q(n-1) + 1$. □

For all parameters β discussed in this paper, $\beta(G) \le n$ (where n is the order of G). Therefore, $\beta^{\times}(r; n) \le n^r$. For tree-width and its variants largeur d'arborescence, path-width and proper path-width, this upper bound is essentially the best we can do.

Theorem 4.2. *Let $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$. For all $r \ge 2$, $\beta^{\times}(r; n) = \beta^{\times}_{\mathrm{nd}}(r; n) = n^r - o(n^r)$.*

Proof. Since $\mathrm{ppw}(G) \ge \mathrm{pw}(G) \ge \mathrm{la}(G) \ge \mathrm{tw}(G)$ for all G, it is enough to prove $\mathrm{tw}^{\times}(r; n) = n^r - o(n^r)$.

By Observation 1.10, $\mathrm{tw}^{\times}(r; n) = \mathrm{tw}^{\times}_{\mathrm{nd}}(r; n)$. Theorem 2.1 shows that $\mathrm{tw}_{\mathrm{nd}}(r; n) = rn - o(n)$, since when n is large enough such that $\mathrm{tw}(r; n) \ge rn - 0.5n$, the value of $\mathrm{tw}(r; n)$ can be achieved only by a non-degenerate decomposition (because $\mathrm{tw}(G) \le n-1$). Fix ϵ with $0 < \epsilon < 1$. When n is large enough, there is a non-degenerate r-decomposition $G_1, \ldots, G_r$ such that

$$\mathrm{tw}(G_1) + \cdots + \mathrm{tw}(G_r) \ge (r - \epsilon)n.$$

Applying Lemma 4.1 with $\sigma = (r-\epsilon)n$, we get $q = \left\lfloor \frac{(r-\epsilon)n-r}{n-1} \right\rfloor = \left\lfloor r - \frac{\epsilon n}{n-1} \right\rfloor = r-1$ for n large enough, and $\rho = (r-\epsilon)n - (r-1)n = (1-\epsilon)n$, so

$$\mathrm{tw}(G_1)\cdots\mathrm{tw}(G_r) \geq (1-\epsilon)n^r.$$

Since $\epsilon > 0$ can be taken arbitrarily small, $\mathrm{tw}^{\times}_{\mathrm{nd}}(r;n) = n^r - o(n^r)$. □

Theorem 4.3. *Let $\beta \in \{\xi, \nu, \mu, \eta\}$. For $r \geq 2$, and any $n \geq 2\sqrt{r}$,*

$$t^{-r}n^r - o(n^r) \leq \beta^{\times}(r;n) \leq r^{-\frac{r}{2}}n^r + o(n^r),$$

where $t := \lceil \mathrm{trt}(r) \rceil = \left\lceil \sqrt{2r+\frac{1}{4}} - \frac{1}{2} \right\rceil$.

Proof. The upper bound comes immediately from Theorem 2.2 and the AM-GM inequality. To see the lower bound, we build a decomposition as follows. Let t be the minimum integer such that $r \leq \frac{t(t+1)}{2}$, i.e., $t = \lceil \mathrm{trt}(r) \rceil$. Partition the vertex set into t parts $V_1, V_2, \ldots, V_t$ with $|V_i| \geq \left\lfloor \frac{n}{t} \right\rfloor$ for all i. Consider H_i as the subgraph with all edges in V_i, and $H_{i,j}$ as the subgraph with all edges between V_i and V_j. We have $\nu(H_i) \geq \eta(H_i) - 1 \geq \left\lfloor \frac{n}{t} \right\rfloor - 1$ and $\nu(H_{i,j}) \geq \eta(H_{i,j}) - 1 \geq \left\lfloor \frac{n}{t} \right\rfloor$ (and similarly for μ). Take r subgraphs out of those H_i and $H_{i,j}$, then merge all remaining edges to one of the subgraphs. This builds a r-decomposition with product at least

$$\left(\left\lfloor \frac{n}{t} \right\rfloor - 1 \right)^r.$$

Therefore, $\beta^{\times}(r;n) \geq t^{-r}n^r - o(n^r)$. □

5 Nordhaus-Gaddum Product Lower Bounds

For $\beta \in \{\eta, \mu, \nu, \xi, \mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$ we show that the growth rate of $\underline{\beta}^{\times}_{\mathrm{nd}}(r;n)$ is $\Theta(n)$.

Recall that for $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$, $\beta(\overline{K_n}) = 0$, so $\underline{\beta}^{\times}(r;n) = 0$; therefore, for these parameters we focus on the non-degenerate case. We first prove a technical result that allows us to convert a sum lower bound to a product lower bound for non-degenerate decompositions.

Theorem 5.1. *Suppose that for $r \geq 2$ and every graph G on n vertices that has an edge, $1 \leq \beta(G) \leq n$ and $\underline{\beta}_{\mathrm{nd}}(r;n) < n + r - 1$. Then*

$$\underline{\beta}_{\mathrm{nd}}(r;n) - r + 1 \leq \underline{\beta}^{\times}_{\mathrm{nd}}(r;n)$$

This formula also applies to $\underline{\beta}^{\times}(r;n)$ if the hypotheses are satisfied for all graphs G (without the restriction of having an edge).

Proof. Let $G_1, \ldots, G_r$ be a non-degenerate r-decomposition that achieves $\underline{\beta}^{\times}_{\mathrm{nd}}(r;n)$. Pick $1 \leq a_i \leq \beta(G_i)$ such that $\sum_{i=1}^{r} a_i = \underline{\beta}_{\mathrm{nd}}(r;n) < n + r - 1$. Now apply Lemma 4.1 with $\sigma = \underline{\beta}_{\mathrm{nd}}(r;n)$. Then $q = \left\lfloor \frac{\sigma - r}{n-1} \right\rfloor = 0$ and $\rho = \sigma - r + 1$. Therefore,

$$n^q \rho = \sigma - r + 1 \leq \prod_{i=1}^{r} a_i \leq \underline{\beta}^{\times}_{\mathrm{nd}}(r;n).$$

When $\beta(G) \geq 1$ for all G, the same argument works for $\underline{\beta}^{\times}(r;n)$. □

Theorem 5.2. *Let $\beta \in \{\mathrm{tw}, \mathrm{la}, \mathrm{pw}, \mathrm{ppw}\}$.*

(i) For $n \geq 4$,
$$\underline{\beta}^{\times}_{\mathrm{nd}}(2; n) = n - 3.$$

(ii) For a fixed $r \geq 3$ and n large enough,
$$\frac{n}{2} - r + 1 \leq \underline{\beta}^{\times}_{\mathrm{nd}}(r; n) \leq n - 2r + 1.$$

Proof. By the non-degenerate r-decomposition of K_n into $r - 1$ paths and one large piece in Theorem 3.7,
$$\underline{\mathrm{ppw}}^{\times}_{\mathrm{nd}}(r; n) \leq n - 2r + 1.$$
Note that $\underline{\mathrm{tw}}_{\mathrm{nd}}(r; n) \geq \underline{\mathrm{tw}}(r; n)$ by definition. By [7, 13], $\underline{\mathrm{tw}}(2; n) \geq n - 2$, and for $r \geq 3$ and n large enough, $\underline{\mathrm{tw}}(r; n) \geq \frac{n}{2}$ by Corollary 3.5. Consequently, Theorem 5.1 implies $\underline{\mathrm{tw}}^{\times}_{\mathrm{nd}}(2; n) \geq n - 3$, and $\underline{\mathrm{tw}}^{\times}_{\mathrm{nd}}(r; n) \geq \frac{n}{2} - r + 1$ for $r \geq 3$ and n large enough. □

Note that Theorem 5.2(i) was established in [12] but non-degeneracy was implicitly assumed and should have been stated. Next we consider the Hadwiger number. Since $\eta(G) = 1$ if and only if G has no edges, both the general and non-degenerate decompositions are of interest, and the product lower bounds have different values in the case $r = 2$.

Remark 5.3. It is known [16] that $\underline{\eta}^{\times}_{\mathrm{nd}}(2; n) \geq \left\lceil \frac{3n-5}{2} \right\rceil$. If one of the two parts has no edge, then the decomposition becomes K_n and $\overline{K_n}$, and the product is n. When $n \geq 4$, $n \leq \left\lceil \frac{3n-5}{2} \right\rceil$, so $\underline{\eta}^{\times}(2; n) = n$ (because the decomposition $K_n, \overline{K_n}$ also provides an upper bound). So $\underline{\eta}^{\times}_{\mathrm{nd}}(2; n) > \underline{\eta}^{\times}(2; n)$ for $n \geq 5$. By checking small cases, we see that $\underline{\eta}^{\times}(2; n) = n$ for all n.

For a graph G on n vertices, Balogh and Kostochka, building on work of Duchet and Meyniel [6] and Fox [9], showed in [2] that
$$0.513n \leq \eta(G)\omega(\overline{G}).$$

Theorem 5.4. *For all $r \geq 2$ and $n \geq 1$,*
$$(0.513)^{r-2} n \leq \underline{\eta}^{\times}(r; n) \leq n.$$

Proof. The upper bound is achieved by one complete graph and $r - 1$ empty graphs.

We prove the lower bound by induction. For base case $r = 2$, we already know $\underline{\eta}^{\times}(2; n) = n$. Assuming $\underline{\eta}^{\times}(r - 1; n) \geq (0.513)^{r-3} n$, we consider a r-decomposition $G_1, \ldots, G_r$. Since $\eta(G_1)\omega(\overline{G_1}) \geq 0.513n$, there is a clique in $\overline{G_1}$ on the vertex set W with $|W| \geq 0.513 \frac{n}{\eta(G_1)}$. Now
$$\begin{aligned} \eta(G_1) \cdots \eta(G_r) &\geq \eta(G_1)\eta(G_2[W]) \cdots \eta(G_r[W]) \\ &\geq \eta(G_1) \cdot (0.513)^{r-3} |W| \geq (0.513)^{r-2} n, \end{aligned}$$
since $G_2[W], \ldots, G_r[W]$ form an $(r - 1)$-decomposition of the clique on W. □

Remark 5.5. It is known that $\underline{\chi}^\times(r;n) = n$ for all r and n [10, p. 277]. On the other hand, Hadwiger's conjecture states that $\eta(G) \geq \chi(G)$ for all graph G. Therefore, Hadwiger's conjecture implies $\underline{\eta}^\times(r;n) \geq n$, and thus implies Conjecture 1.7, so any counterexample to Conjecture 1.7 would disprove Hadwiger's conjecture.

Corollary 5.6. *For all $r \geq 2$ and $n \geq 2r$,*

$$(0.513)^{r-2}n \leq \underline{\eta}^\times_{\rm nd}(r;n) \leq 2^{r-1}(n-2r+2).$$

Proof. The lower bound is by Theorem 5.4 and the fact $\underline{\eta}^\times_{\rm nd}(r;n) \geq \underline{\eta}^\times(r;n)$. On the other hand, let $P_1, P_2, \ldots, P_r$ be the r-decomposition in Theorem 3.3. Recall that the Hadwiger number of a path is 2 whenever it has more than one vertex. Since P_i is a path for $i = 1, \ldots, r-1$ and $\eta(P_r) \leq \text{ppw}(P_r) + 1 = n - 2r + 2$, we have $\underline{\eta}^\times_{\rm nd}(r;n) \leq 2^{r-1}(n-2r+2)$. □

A peculiarity of the parameters ξ, ν, and μ is that $\beta(P_n) = \beta(\overline{K_n}) = 1$ for $\beta \in \{\xi, \nu, \mu\}$ (and $n \geq 2$ in the case of μ). Thus $\underline{\beta}^\times$ is not optimized on $\overline{K_n}$ for these parameters, and we focus on the non-degenerate versions.

Corollary 5.7. *Let $\beta \in \{\nu, \xi, \mu\}$. For $r \geq 2$ and $n \geq 2r$,*

$$\frac{n}{2^{2r-2}} \leq \underline{\beta}^\times_{\rm nd}(r;n) \leq n - 2r + 1.$$

Proof. For $\beta \in \{\nu, \xi, \mu\}$, $\eta(G) - 1 \leq \beta(G) \leq \text{ppw}(G)$ when G has an edge. The result then follows from Theorem 5.2 and Corollary 5.6, since $\beta(G) \geq \eta(G) - 1 \geq \frac{1}{2}\eta(G)$ and $\underline{\eta}^\times_{\rm nd}(r;n) \geq (0.513)^{r-2}n \geq \frac{n}{2^{r-2}}$. □

Acknowledgments

This research began while the authors were general members in residence at the Institute for Mathematics and its Applications, and they thank the IMA both for financial support and for a wonderful research environment.

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