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On the distance spectra of graphs

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ABSTRACT

The distance matrix of a graph G is the matrix containing the pairwise distances between vertices. The distance eigenvalues of G are the eigenvalues of its distance matrix and they form the distance spectrum of G. We determine the distance spectra of double odd graphs and Doob graphs, completing the determination of distance spectra of distance regular graphs having exactly one positive distance eigenvalue. We

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characterize strongly regular graphs having more positive than negative distance eigenvalues. We give examples of graphs with few distinct distance eigenvalues but lacking regularity properties. We also determine the determinant and inertia of the distance matrices of lollipop and barbell graphs.

Keywords:
Distance matrix
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Barbell graph
Distance spectrum
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Optimistic graph
Determinant
Inertia
Graph

1. Introduction

The *distance matrix* $\mathcal{D}(G) = [d_{ij}]$ of a graph G is the matrix indexed by the vertices $\{v_1, \ldots, v_n\}$ of G where $d_{ij} = d(v_i, v_j)$ is the *distance* between the vertices v_i and v_j, i.e., the length of a shortest path between v_i and v_j. Distance matrices were introduced in the study of a data communication problem in [17]. This problem involves finding appropriate addresses so that a message can move efficiently through a series of loops from its origin to its destination, choosing the best route at each switching point. Recently there has been renewed interest in the loop switching problem [14]. There has also been extensive work on distance spectra (eigenvalues of distance matrices); see [3] for a recent survey.

In [20], the authors classify the distance regular graphs having exactly one positive distance eigenvalue. Such graphs are directly related to a metric hierarchy for finite connected graphs (and more generally, for finite distance spaces, see [4,12,13,19]), which makes these graphs particularly interesting. In Section 3 we find the distance spectra of Doob graphs and double odd graphs, completing the determination of distance spectra of distance regular graphs that have exactly one positive distance eigenvalue. In Section 2 we characterize strongly regular graphs having more positive than negative distance eigenvalues in terms of their parameters, generalizing results in [7], and apply this characterization to show several additional infinite families of strongly regular graphs have this property.

Section 4 contains examples of graphs with specific properties and a small number of distinct distance eigenvalues. Answering a question in [5], we provide a construction for a family of connected graphs with arbitrarily large diameter that have no more than 5 distinct distance eigenvalues but are not distance regular (Example 4.2). We exhibit a family of graphs with arbitrarily many distinct degrees but having exactly 5 distinct distance eigenvalues (Example 4.5). Finally, we give two lower bounds for the number of distinct distance eigenvalues of a graph. The first bound is for a tree, in terms of its diameter, and the second is for any graph in terms of the zero forcing number of its complement.

In Persi Diaconis' talk on distance spectra at the "Connections in Discrete Mathematics: A celebration of the work of Ron Graham" [14], he suggested it would be worthwhile to study the distance matrix of a clique with a path adjoined (sometimes called a lollipop graph), and in Section 5 we determine determinants and inertias of these graphs, of barbell graphs, and of generalized barbell graphs (a family that includes both lollipops and barbells). The remainder of this introduction contains definitions and notation used throughout.

All graphs considered here are connected, simple, undirected, and finite of order at least two. Let G be a graph. The maximum distance between any two vertices in G is called the *diameter* of G and is denoted by $\operatorname{diam}(G)$. Two vertices are adjacent in the *complement* of G, denoted by $\overline{G}$, if and only if they are nonadjacent in G. Let $\mathcal{A}(G)$ denote the *adjacency matrix* of G, that is, $\mathcal{A}(G) = [a_{ij}]$ is the matrix indexed by the vertices $\{v_1, \ldots, v_n\}$ of G where $a_{ij} = 1$ if $\{v_i, v_j\} \in E(G)$ and is 0 otherwise. The $n \times n$ all ones matrix is denoted by J and the all ones vector by $\mathbb{1}$. A graph G is *regular* if every vertex has the same degree, say k; equivalently, $\mathcal{A}(G)\mathbb{1} = k\mathbb{1}$; observe that k is the spectral radius of $\mathcal{A}(G)$.

Since $\mathcal{D}(G)$ is a real symmetric matrix, its eigenvalues, called *distance eigenvalues* of G, are all real. The spectrum of $\mathcal{D}(G)$ is denoted by $\operatorname{spec}_{\mathcal{D}}(G) := \{\rho, \theta_2, \ldots, \theta_n\}$ where ρ is the spectral radius, and is called the *distance spectrum* of the graph G.

The *inertia* of a real symmetric matrix is the triple of integers (n_+, n_0, n_-), with the entries indicating the number of positive, zero, and negative eigenvalues, respectively (counting multiplicities). Note that the order (n_+, n_0, n_-), while customary in spectral graph theory, is nonstandard in linear algebra, where it is more common to use (n_+, n_-, n_0). The spectrum of a matrix can be written as a multiset (with duplicates as needed), or as a list of distinct values with the exponents in parentheses indicating multiplicity (the exponent 1 is often omitted). In some formulas for spectra, the exponent of a number listed may be zero in a degenerate case, indicating that the number is not in fact an eigenvalue.

2. Strongly regular graphs

A k-regular graph G of order n is *strongly regular* with parameters (n, k, λ, μ) if every pair of adjacent vertices has λ common neighbors and every pair of distinct nonadjacent vertices has μ common neighbors. For $n \geq 2$, the complete graph K_n is strongly regular with parameters $(n, n-1, n-2, 0)$. For a strongly regular graph with parameters (n, k, λ, μ), $\mu = 0$ is equivalent to G being $\frac{n}{k+1}$ copies of K_{k+1}, so we assume $\mu > 0$ or G is complete, and thus G has diameter at most 2. There is a well known connection between the adjacency matrix of a graph of diameter at most 2 and its distance matrix that was exploited in [7].

Remark 2.1. A real symmetric matrix commutes with J if and only if it has constant row sum. Suppose A commutes with J and ρ is the constant row sum of A, so J and A have a common eigenvector of $\mathbb{1}$. Since eigenvectors of real symmetric matrices corresponding to distinct eigenvalues are orthogonal, every eigenvector of A for an eigenvalue other than ρ is an eigenvector of J for eigenvalue 0.

Now suppose G is a graph that has diameter at most 2. Then $\mathcal{D}(G) = 2(J-I) - \mathcal{A}(G)$. In addition suppose that G is regular, so $\mathcal{A}(G)$ commutes with J. Thus $\operatorname{spec}_{\mathcal{D}}(G) = \{2n-2-\rho\} \cup \{-\lambda-2 : \lambda \in \operatorname{spec}(\mathcal{A}(G)) \text{ and } \lambda \neq \rho\}$.

Let G be a strongly regular graph with parameters (n, k, λ, μ). It is known that the (adjacency) eigenvalues of G are $\rho = k$ of multiplicity 1, $\theta := \frac{1}{2}(\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)})$ of multiplicity $m_\theta := \frac{1}{2}(n-1-\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}})$, and $\tau := \frac{1}{2}(\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)})$ of multiplicity $m_\tau := \frac{1}{2}(n-1+\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}})$ [16, Chapter 10].[1] Thus the distance eigenvalues of G are

$$\begin{aligned}
&\rho_{\mathcal{D}} := 2(n-1)-k \qquad \text{of multiplicity } 1\\
&\theta_{\mathcal{D}} := -\frac{1}{2}\left(\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right)-2\\
&\qquad \text{of multiplicity } m_\theta = \frac{1}{2}\left(n-1-\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right)\\
&\tau_{\mathcal{D}} := -\frac{1}{2}\left(\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right)-2\\
&\qquad \text{of multiplicity } m_\tau = \frac{1}{2}\left(n-1+\frac{2k+(n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right).
\end{aligned}$$

For a derivation of these values using quotient matrices, see [5, p. 262].

2.1. Optimistic strongly regular graphs

A graph is *optimistic* if it has more positive than negative distance eigenvalues. Graham and Lovász raised the question of whether optimistic graphs exist (although they did not use the term). This question was answered positively by Azarija in [7], where the term 'optimistic' was introduced. A strongly regular graph is a *conference graph* if $(n, k, \lambda, \mu) = (n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$. In [7] it is shown that conference graphs of order at least 13 are optimistic and also that the strongly regular graphs with parameters $(m^2, 3(m-1), m, 6)$ are optimistic for $m \geq 5$. Additional examples of optimistic strongly regular graphs, such as the Hall–Janko graph with parameters (100, 36, 14, 12), and examples of optimistic graphs that are not strongly regular are also presented there.

[1] Although [16] requires $G \neq K_n$, these formulas do work for K_n (producing $m_\theta = 0$).

Theorem 2.2. *Let G be a strongly regular graph with parameters (n, k, λ, μ). The graph G is optimistic if and only if $\tau_{\mathcal{D}} > 0$ and $m_\tau \geq m_\theta$. That is, G is optimistic if and only if*

$$\mu - \frac{2k}{n-1} \leq \lambda < \frac{-4 + \mu + k}{2}.$$

Proof. Observe that $\theta_{\mathcal{D}} < 0$. Thus G is optimistic if and only if $\tau_{\mathcal{D}} > 0$ and $m_\tau \geq m_\theta$. Simple algebra shows that $0 < \tau_{\mathcal{D}}$ is equivalent to $\lambda < \frac{-4+\mu+k}{2}$:

$$\begin{aligned} 0 &< -2 - \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right) \\ 4 + \lambda - \mu &< \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}. \end{aligned}$$

There are two cases. First assume $4 + \lambda - \mu \geq 0$, so

$$\begin{aligned} (\lambda-\mu)^2 + 8(\lambda-\mu) + 16 &< (\lambda-\mu)^2 + 4(k-\mu) \\ \lambda &< \frac{-4+\mu+k}{2}. \end{aligned}$$

Now assume $4 + \lambda - \mu < 0$, or equivalently, $\lambda - \mu < -4$. For any strongly regular graph, $k \geq \lambda + 1$. Thus

$$\begin{aligned} (\lambda-\mu) + (\lambda-k) &< -4 \\ \lambda &< \frac{-4+\mu+k}{2}. \end{aligned}$$

It is well known [16, p. 222] (and easy to see) that

$$m_\tau - m_\theta = \frac{2k + (n-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}. \tag{1}$$

The denominator is always positive, and thus $m_\tau \geq m_\theta$ if and only if $\lambda \geq \mu - \frac{2k}{n-1}$. $\square$

There are several additional families of strongly regular graphs for which the conditions in Theorem 2.2 hold.

Corollary 2.3. *A strongly regular graph with parameters (n, k, μ, μ) is optimistic if and only if $k > \mu + 4$.*

Proof. By Theorem 2.2, $\lambda = \mu$ implies $m_\tau > m_\theta$, and $\tau_{\mathcal{D}} > 0$ is equivalent to

$$0 < \frac{-4+\mu+k}{2} - \lambda = \frac{1}{2}(k - \mu - 4). \qquad \square$$

The family of symplectic graphs is defined using subspaces of a vector space over a field with a finite number of elements. Let $\mathbb{F}_q$ be the field with q elements and consider as vertices of $Sp(2m,q)$ the one dimensional subspaces of $\mathbb{F}_q^{2m}$ for $m \geq 2$; let $\langle \mathbf{x} \rangle$ denote the subspace generated by $\mathbf{x}$. The *alternate matrix* of order m over $\mathbb{F}_q$ is the $2m \times 2m$ matrix $A_m = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with m copies of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The vertices $\langle \mathbf{x} \rangle$ and $\langle \mathbf{y} \rangle$ are adjacent in $Sp(2m,q)$ if $\mathbf{x}^T A_m \mathbf{y} \neq 0$. See [16, Section 8.11] for $Sp(2m,2)$ and [24] for more general q. It is known that the symplectic graph $Sp(2m,q)$ is a strongly regular graph with parameters

$$(n,k,\lambda,\mu) = \left(\frac{q^{2m}-1}{q-1},\, q^{2m-1},\, q^{2m-2}(q-1),\, q^{2m-2}(q-1) \right)$$

(see [16, Section 10.12] for $q = 2$ or [24, Theorem 2.1] for more general q). The next result is immediate from Corollary 2.3.

Corollary 2.4. *The symplectic graphs $Sp(2m,q)$ are optimistic for every q and m except $q = 2$ and $m = 2$.*

There are additional families of optimistic strongly regular graphs with parameters (n,k,μ,μ). One example is the family $O_{2m+1}(3)$ on one type of nonisotropic points, which has parameters

$$\left(\frac{3^m\,(e+3^m)}{2}, \frac{3^{m-1}\,(3^m-e)}{2}, \frac{3^{m-1}\left(3^{m-1}-e\right)}{2}, \frac{3^{m-1}\left(3^{m-1}-e\right)}{2} \right)$$

with $m \geq 2$ and $e = \pm 1$ [9].

The more common definition of a conference graph is a strongly regular graph with $m_\theta = m_\tau$; this is equivalent to the definition given earlier as $(n,k,\lambda,\mu) = (n, \frac{n-1}{2}, \frac{n-5}{4}, \frac{n-1}{4})$ [16, Lemma 10.3.2].

Theorem 2.5. *Let G be a strongly regular graph with parameters (n,k,λ,μ). Both G and $\overline{G}$ are optimistic if and only if G is a conference graph and $n \geq 13$.*

Proof. The parameters of $\overline{G}$ are $(n,\ \bar{k} := n-k-1,\ \bar{\lambda} := n-2-2k+\mu,\ \bar{\mu} := n-2k+\lambda)$. By applying Theorem 2.2 to G and $\overline{G}$,

$$\lambda \geq \mu - \frac{2k}{n-1} \tag{2}$$

and

$$\bar{\lambda} \geq \bar{\mu} - \frac{2\bar{k}}{n-1}$$

$$n-2-2k+\mu \geq n-2k+\lambda - \frac{2(n-k-1)}{n-1}$$

$$-2 + \frac{2(n-k-1)}{n-1} + \mu \geq \lambda$$

$$\mu - \frac{2k}{n-1} \geq \lambda. \tag{3}$$

By comparing (2) and (3), we see that $\lambda = \mu - \frac{2k}{n-1}$, which by (1) implies $m_\tau - m_\theta = 0$. Thus G is a conference graph. It is shown in [7] that conference graphs are optimistic if and only if the order is at least 13. □

2.2. Strongly regular graphs with exactly one positive distance eigenvalue

Distance regular graphs (see Section 3 for the definition) having exactly one positive distance eigenvalue were studied in [20]; strongly regular graphs are distance regular. Here we make some elementary observations about strongly regular graphs with exactly one positive distance eigenvalue that will be used in Section 3.

Proposition 2.6. *Let G be a strongly regular graph with parameters (n, k, λ, μ), and (adjacency) eigenvalues k, θ, and τ. Then $G = C_5$, $G = K_n$, or $\tau \leq -2$. Thus G has exactly one positive distance eigenvalue if and only if $G = C_5$, $G = K_n$, or $\tau = -2$.*

Proof. From the formulas for the eigenvalues of a strongly regular graph, $\theta > 0$ and $\tau < 0$. It is known that if a strongly regular graph G is not a conference graph, then τ and θ are integers [16, Lemma 10.3.3]. If $\tau = -1$, then $k = \lambda + 1$ (this follows from $\tau^2 - (\lambda - \mu)\tau - (k - \mu) = 0$ [16, p. 219]), implying $G = K_n$. Thus G is a conference graph, $G = K_n$, or $\tau \leq -2$. Since $\tau = \frac{1}{2}(-\sqrt{n} - 1)$ for a conference graph on n vertices, every conference graph on at least 9 vertices has $\tau \leq -2$; the conference graph on 5 vertices is C_5.

Since $\theta > 0$, $\tau < 0$, $\theta_{\mathcal{D}} = -2 - \theta$, and $\tau_{\mathcal{D}} = -2 - \tau$, G has exactly one positive eigenvalue if and only if $\tau_{\mathcal{D}} \leq 0$ if and only if $\tau \geq -2$. Thus, the fact that G has only one positive eigenvalue implies $G = C_5$, $G = K_n$, or $\tau = -2$. Since K_n and C_5 each have exactly one positive distance eigenvalue, G has exactly one positive distance eigenvalue if and only if $\tau = -2$, $G = C_5$, or $G = K_n$. □

Observation 2.7. There are several well known families of strongly regular graphs having $\tau = -2$, and thus having one exactly positive distance eigenvalue. Examples of such graphs and their distance spectra include:

1. The cocktail party graphs $CP(m)$ are complete multipartite graphs $K_{2,2,\ldots,2}$ on m partite sets of order 2; $CP(m)$ is a strongly regular graph with parameters $(2m, 2m-2, 2m-4, 2m-2)$ and has distance spectrum $\{2m, 0^{(m-1)}, -2^{(m)}\}$.
2. The line graph $L(K_m)$ with parameters $(\frac{m(m-1)}{2}, 2m-4, m-2, 4)$ has distance spectrum $\{(m-1)(m-2), 0^{(\frac{m(m-3)}{2})}, (2-m)^{(m-1)}\}$.

3. The line graph $L(K_{m,m})$ with parameters $(m^2, 2m-2, m-2, 2)$ has distance spectrum $\{2m(m-1), 0^{((m-1)^2)}, (-m)^{(2(m-1))}\}$.

3. Distance regular graphs having exactly one positive distance eigenvalue

Let $i, j, k \geq 0$ be integers. The graph $G = (V, E)$ is called *distance regular* if for any choice of $u, v \in V$ with $d(u, v) = k$, the number of vertices $w \in V$ such that $d(u, w) = i$ and $d(v, w) = j$ is independent of the choice of u and v. Distance spectra of several families of distance regular graphs were determined in [5]. In this section we complete the determination of the distance spectra of all distance regular graphs having exactly one positive distance eigenvalue, as listed in [20, Theorem 1]. For individual graphs, it is simply a matter of computation, but for infinite families the determination is more challenging. The infinite families in [20, Theorem 1] are (with numbering from that paper):

(I) cocktail party graphs $CP(m)$,
(X) cycles C_n (called polygons in [20]),
(VII) Hamming graphs $H(d, n)$,
(IV) halved cubes $\frac{1}{2}Q_d$,
(VIII) Doob graphs $D(d, n)$,
(V) Johnson graphs $J(n, r)$, and
(XI) double odd graphs $DO(r)$.

First we summarize the known distance spectra of these infinite families. For a strongly regular graph, it is easy to determine the distance spectrum and we have listed the distance spectra of cocktail party graphs in Observation 2.7. The distance spectra of cycles are determined in [15], and in [5] are presented in the following form: For odd $n = 2p + 1$, $\operatorname{spec}_{\mathcal{D}}(C_n) = \left\{\frac{n^2-1}{4}, (-\frac{1}{4}\sec^2(\frac{\pi j}{n}))^{(2)}, j = 1, \dots, p\right\}$, and for even $n = 2p$ the distance eigenvalues are $\frac{n^2}{4}, 0^{(p-1)}, (-\csc^2(\frac{\pi(2j-1)}{n}))^{(2)}, j = 1, \dots, \left\lfloor \frac{p}{2} \right\rfloor$ and -1 if p is odd.

Hamming graphs, whose distance spectra are determined in [18], and halved cubes, whose distance spectra are determined in [6], are discussed in Section 3.1 together with Doob graphs; the distance spectra of Doob graphs are established there. Johnson graphs, whose distance spectra are determined in [5], are discussed in Section 3.2 together with double odd graphs, and the distance spectra of double odd graphs are established there. Thus now all the infinite families in [20, Theorem 1] have their distance spectra determined.

For completeness we list the distance spectra (some of which are known) for the individual graphs having exactly one positive distance eigenvalue; these are easily computed and we provide computational files in *Sage* [22]. Definitions of these graphs can be found in [3].

Proposition 3.1. *The graphs listed in [20, Theorem 1] that have exactly one positive distance eigenvalue and are not in one of the infinite families, together with their distance spectra, are:*

(II) *the Gosset graph;* $\{84, 0^{(48)}, (-12)^{(7)}\}$,
(III) *the Schläfli graph;* $\{36, 0^{(20)}, (-6)^{(6)}\}$,
(VI) *the three Chang graphs (all the same spectra);* $\{42, 0^{(20)}, (-6)^{(7)}\}$,
(IX) *the icosahedral graph;* $\left\{18, 0^{(5)}, (-3+\sqrt{5})^{(3)}, (-3-\sqrt{5})^{(3)}\right\}$,
(XII) *the Petersen graph;* $\{15, 0^{(4)}, (-3)^{(5)}\}$,
(XIII) *the dodecahedral graph;* $\left\{50, 0^{(9)}, (-7+3\sqrt{5})^{(3)}, (-2)^{(4)}, (-7-3\sqrt{5})^{(3)}\right\}$.

A graph G is *transmission regular* if $\mathcal{D}(G)\mathbb{1} = \rho\mathbb{1}$ (where ρ is the constant row sum of $\mathcal{D}(G)$). Any distance regular graph is transmission regular. Here we present some tools for transmission regular graphs and matrices constructed from distance matrices of transmission regular graphs. We first define the *cartesian product* of two graphs: For graphs $G = (V, E)$ and $G' = (V', E')$ define the graph $G \,\square\, G'$ to be the graph whose vertex set is the cartesian product $V \times V'$ and where two vertices (u, u') and (v, v') are adjacent if ($u = v$ and $\{u', v'\} \in E'$) or ($u' = v'$ and $\{u, v\} \in E$). The next theorem is stated for distance regular graphs in [18], but as noted in [5] the proof applies to transmission regular graphs.

Theorem 3.2. *(See [18, Theorem 2.1].) Let G and G' be transmission regular graphs with* $\operatorname{spec}_{\mathcal{D}}(G) = \{\rho, \theta_2, \ldots, \theta_n\}$ *and* $\operatorname{spec}_{\mathcal{D}}(G') = \{\rho', \theta'_2, \ldots, \theta'_{n'}\}$*. Then*

$$\operatorname{spec}_{\mathcal{D}}(G \,\square\, G') = \{n'\rho + n\rho'\} \cup \{n'\theta_2, \ldots, n'\theta_n\} \cup \{n\theta'_2, \ldots, n\theta'_{n'}\} \cup \{0^{((n-1)(n'-1))}\}.$$

Lemma 3.3. *Let D be an $n \times n$ irreducible nonnegative symmetric matrix that commutes with J. Suppose*

$$M = \begin{bmatrix} a_e D + b_e J + c_e I & a_o D + b_o J + c_o I \\ a_o D + b_o J + c_o I & a_e D + b_e J + c_e I \end{bmatrix},$$

$a_e, a_o, b_e, b_o, c_e, c_o \in \mathbb{R}$, *and* $\operatorname{spec}(D) = \{\rho, \theta_2, \ldots, \theta_n\}$ *(where ρ is the row sum so $D\mathbb{1} = \rho\mathbb{1}$). Then*

$$\begin{aligned}\operatorname{spec}(M) = \{&(a_e + a_o)\rho + (b_e + b_o)n + (c_e + c_o), (a_e - a_o)\rho + (b_e - b_o)n + (c_e - c_o)\} \\ &\cup \{(a_e + a_o)\theta_i + (c_e + c_o) : i = 2, \ldots, n\} \\ &\cup \{(a_e - a_o)\theta_i + (c_e - c_o) : i = 2, \ldots, n\},\end{aligned}$$

where the union is a multiset union.

Proof. Since D commutes with J, $\mathbb{1}$ is an eigenvector of D for eigenvalue ρ. Thus $\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix}$ is an eigenvector for $(a_e + a_o)\rho + (b_e + b_o)n + (c_e + c_o)$ and $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$ is an eigenvector for $(a_e - a_o)\rho + (b_e - b_o)n + (c_e - c_o)$. Let $\mathbf{x}_i$ be an eigenvector for θ_i (and assume $\{\mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is linearly independent). Since eigenvectors for distinct eigenvalues are orthogonal, $\mathbf{x}_i \perp \mathbb{1}$, and thus $J\mathbf{x}_i = 0$. Define $\mathbf{y}_i = \begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_i \end{bmatrix}$ and $\mathbf{z}_i = \begin{bmatrix} \mathbf{x}_i \\ -\mathbf{x}_i \end{bmatrix}$. Then $\mathbf{y}_i$ is an eigenvector of M for eigenvalue $(a_e + a_o)\theta_i + (c_e + c_o)$ and $\mathbf{z}_i$ is an eigenvector of M for eigenvalue $(a_e - a_o)\theta_i + (c_e - c_o)$. □

3.1. Hamming graphs, halved cubes, and Doob graphs

For $n \geq 2$ and $d \geq 1$, the Hamming graph $H(d, n)$ has vertex set consisting of all d-tuples of elements taken from $\{0, \ldots, n-1\}$, with two vertices adjacent if and only if they differ in exactly one coordinate; $H(d, n)$ is equal to $K_n \,\square \cdots \square\, K_n$ with d copies of K_n. In [18] it is shown that the distance spectrum of the Hamming graph $H(d, n)$ is

$$\operatorname{spec}_{\mathcal{D}}(H(d,n)) = \left\{ dn^{d-1}(n-1), 0^{(n^d - d(n-1)-1)}, (-n^{d-1})^{(d(n-1))} \right\}. \tag{4}$$

Observe that $K_n = H(1, n)$ and the line graph $L(K_{n,n})$ is the Hamming graph $H(2, n)$.

The Hamming graph $H(d, 2)$ is also called the dth hypercube and denoted by Q_d. The halved cube $\frac{1}{2}Q_d = (V, E)$ is constructed from Q_d as follows: The set of vertices V of $\frac{1}{2}Q_d$ is equal to the subset of vertices of Q_d that have an even number of ones. Two vertices of $\frac{1}{2}Q_d$ are adjacent if they are at distance two in Q_d, or equivalently if they differ in exactly two coordinates. It is known that the halved cube is distance regular and has exactly one positive distance eigenvalue [20].

For any graph $G = (V, E)$, the square is the graph $G^2 = (V, E_2)$ with the same vertex set and edge set $E_2 = \{\{u, v\} : d(u, v) \leq 2 \text{ and } u \neq v\}$.

Theorem 3.4. *(See [6, Theorem 2.3].) For $d \geq 2$, the distance spectrum of the squared hypercube Q_d^2 is*

$$\left\{ \frac{1}{2} \sum_{i=1}^{d} i \binom{d}{i} + 2^{d-2}, 0^{(2^d - (d+2))}, (-2^{d-2})^{(d+1)} \right\}.$$

Remark 3.5. Appropriate notation shows that the halved cube $\frac{1}{2}Q_d$ is isomorphic to the square of a smaller hypercube, Q_{d-1}^2: Let $V = 2^{d-1}$ be the vertices of Q_{d-1}^2. We can divide V into V_e and V_o, the set of all sequences with an even number of ones and the set of all sequences with an odd number of ones, respectively. For each $x \in V_e$ we let $\bar{x}$ be the sequence of length d by adding a zero to the end of x. For $y \in V_o$, we let $\bar{y}$ be the sequence obtained from y by adding a one to the end of y. Then both $\bar{x}$ and $\bar{y}$ are sequences of length d with even number of ones. They form the vertex set of $\frac{1}{2}Q_d$. Moreover, x and y are adjacent in Q_{d-1}^2 if and only if $\bar{x}$ and $\bar{y}$ are adjacent in $\frac{1}{2}Q_d$.

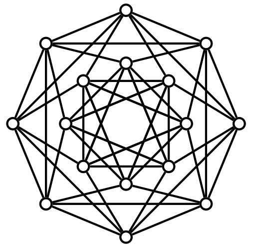

Fig. 1. The Shrikhande graph S.

The next result follows from Theorem 3.4 and Remark 3.5 by simplifying the value of the spectral radius.

Corollary 3.6. *For $d \geq 3$, the distance spectrum of the halved cube $\frac{1}{2}Q_d$ is*

$$\left\{d2^{d-3}, 0^{(2^{d-1}-(d+1))}, (-2^{d-3})^{(d)}\right\}.$$

Remark 3.7. Our original proof of the distance spectrum of a halved cube is combinatorial and much longer than the linear algebraic proof in [6]. It does produce the distance eigenvectors in addition to the distance eigenvalues and can be found in [1].

A Doob graph $D(m,d)$ is the cartesian product of m copies of the Shrikhande graph and the Hamming graph $H(d,4)$. The Shrikhande graph is the graph $S = (V,E)$ where $V = \{0,1,2,3\} \times \{0,1,2,3\}$ and $E = \{\{(a,b),(c,d)\} : (a,b) - (c,d) \in \{\pm(0,1), \pm(1,0), \pm(1,1)\}\}$ (see Fig. 1). The distance spectrum is $\operatorname{spec}_{\mathcal{D}}(S) = \left\{24, 0^{(9)}, (-4)^{(6)}\right\}$ [22].

Theorem 3.8. *The distance spectrum of the Doob graph $D(m,d)$ is*

$$\left\{3(2m+d)4^{2m+d-1}, 0^{(4^{2m+d}-6m-3d-1)}, (-4^{2m+d-1})^{(6m+3d)}\right\}.$$

Proof. Let S_m be the cartesian product of m copies of the Shrikhande graph. Then $\operatorname{spec}_{\mathcal{D}}(D(m,d)) = \operatorname{spec}_{\mathcal{D}}(S_m \,\square\, H(d,4))$. We first show that

$$\operatorname{spec}_{\mathcal{D}}(S_m) = \left\{6m4^{2m-1}, 0^{(16^m-6m-1)}, (-4^{2m-1})^{(6m)}\right\} \tag{5}$$

by induction on m. It is clear that the result holds for $m = 1$. Assume the result holds for $m-1$. Then by induction hypothesis, $\operatorname{spec}_{\mathcal{D}}(S_{m-1}) = \{6(m-1)4^{2m-3}, 0^{(16^{m-1}-6m+5)}, (-4^{2m-3})^{(6m-6)}\}$. By Theorem 3.2, we have

$$\begin{aligned}\operatorname{spec}_{\mathcal{D}}(S_m) &= \operatorname{spec}_{\mathcal{D}}(S_{m-1} \,\square\, S) \\ &= \left\{16 \cdot 6(m-1)4^{2m-3} + 16^{m-1} \cdot 24\right\} \\ &\quad \cup \left\{(16 \cdot 0)^{(16^{m-1}-6m+5)}, (-16 \cdot 4^{2m-3})^{(6m-6)}\right\}\end{aligned}$$

$$\cup \left\{(16^{m-1} \cdot 0)^{(9)}, (-16^{m-1} \cdot 4)^{(6)}\right\} \cup \left\{0^{((16^{m-1}-1)(16-1))}\right\}$$
$$= \left\{6m4^{2m-1}, 0^{(16^m-6m-1)}, (-4^{2m-1})^{(6m)}\right\}.$$

From (4), $\operatorname{spec}_{\mathcal{D}}(H(d,4)) = \left\{3d4^{d-1}, 0^{(4^d-3d-1)}, (-4^{d-1})^{(3d)}\right\}$. Then by (5) and Theorem 3.2, we have

$$\begin{aligned}\operatorname{spec}_{\mathcal{D}}(D(m,d)) &= \operatorname{spec}_{\mathcal{D}}(S_m \,\square\, H(d,4)) \\ &= \left\{4^d 6m4^{2m-1} + 16^m 3d4^{d-1}\right\} \cup \left\{(4^d \cdot 0)^{(16^m-6m-1)}, (-4^d 4^{2m-1})^{(6m)}\right\} \\ &\quad \cup \left\{(16^m \cdot 0)^{(4^d-3d-1)}, (-16^m 4^{d-1})^{(3d)}\right\} \cup \left\{0^{((16^m-1)(4^d-1))}\right\} \\ &= \left\{3(2m+d)4^{2m+d-1}, 0^{(4^{2m+d}-6m-3d-1)}, (-4^{2m+d-1})^{(6m+3d)}\right\}. \qquad \square\end{aligned}$$

3.2. Johnson, Kneser, and double odd graphs

Before defining Johnson graphs, we consider a more general family that includes both Johnson and Kneser graphs. For fixed integers n and r, let $[n] := \{1, \ldots, n\}$ and $\binom{[n]}{r}$ denote the collection of all r-subsets of $[n]$. For fixed integers n, r, i, the graph $J(n; r; i)$ is the graph defined on vertex set $\binom{[n]}{r}$ such that two vertices S and T are adjacent if and only if $|S \cap T| = r - i$.[2] The Johnson graphs $J(n,r)$ are the graphs $J(n; r; 1)$, and they are distance regular. Observe that the line graph $L(K_n)$ is the Johnson graph $J(n,2)$. The distance spectra of Johnson graphs $J(n,r)$ are determined in [5, Theorem 3.6]:

$$\operatorname{spec}_{\mathcal{D}}(J(n,r)) = \left\{s(n,r), 0^{(\binom{n}{r}-n)}, \left(-\frac{s(n,r)}{n-1}\right)^{(n-1)}\right\} \tag{6}$$

where $s(n,r) = \sum_{j=0}^{r} j\binom{r}{j}\binom{n-r}{j}$.

Although they do not necessarily have exactly one positive distance eigenvalue and are not all distance regular, Kneser graphs can be used to construct double odd graphs. The Kneser graph $K(n,r)$ is the graph $J(n; r; r)$. Of particular interest are the *odd graphs* $O(r) = K(2r+1, r)$.

A double odd graph $DO(r)$ is a graph whose vertices are r-element or $(r+1)$-element subsets of $[2r+1]$, where two vertices S and T are adjacent if and only if $S \subset T$ or $T \subset S$, as subsets. Double odd graphs can also be constructed as tensor products of odd graphs. We first define the *tensor product* of two graphs: For graphs $G = (V, E)$ and $G' = (V', E')$ define the graph $G \times G'$ to be the graph whose vertex set is the cartesian product $V \times V'$ and where two vertices (u, u') and (v, v') are adjacent if $\{u, v\} \in E$ and $\{u', v'\} \in E'$. To see that $DO(r) = O(r) \times P_2$, observe that $O(r) \times P_2$ has vertices of

[2] Note that the definition of this family of graphs varies with the source. Here we follow [8], whereas in [16] the graph defined by n, r, and i is what is here denoted by $J(n; r; r-i)$.

two types, $(S_k, 1)$ and $(S_k, 2)$ where S_k is a vertex of $O(r)$. Then there are no edges just between the vertices of the form $(S_k, 1)$, no edges just between the $(S_k, 2)$, and $(S_k, 1) \sim (S_j, 2)$ if and only if $S_k \cap S_j = \emptyset$. Equivalently, $(S_k, 1) \sim (S_j, 2)$ if and only if $S_k \subset \overline{S_j}$. We will work with the representation of $DO(r)$ as $O(r) \times P_2$.

Remark 3.9. Let G be a graph that is not bipartite. Then $G \times P_2$ is a connected bipartite graph and $\mathcal{D}(G \times P_2)$ has the form $\begin{bmatrix} D_e & D_o \\ D_o & D_e \end{bmatrix}$, where D_e and D_o are $n \times n$ nonnegative symmetric matrices with the entries of D_e even and those of D_o odd; all these statements are obvious except the symmetry of D_o. Observe that D_e is the matrix whose i, j-entry is the shortest even distance between vertices v_i and v_j in G, and D_o is the matrix whose i, j-entry is the shortest odd distance between vertices v_i and v_j in G.

It is known [25] that the distance between two vertices S and T in $K(n, r)$ is given by the formula

$$d_K(S, T) = \min\left\{2\left\lceil \frac{r - |S \cap T|}{n - 2r} \right\rceil, 2\left\lceil \frac{|S \cap T|}{n - 2r} \right\rceil + 1\right\},$$

which for the odd graph $O(r)$ is

$$d_O(S, T) = \min\{2(r - |S \cap T|), 2|S \cap T| + 1\}. \tag{7}$$

The distance between two vertices S and T in the Johnson graph $J(n, r)$ is given by the formula

$$d_{J(n,r)}(S, T) = \frac{1}{2}|S \Delta T| = r - |S \cap T|, \tag{8}$$

where Δ is the symmetric difference. Let $d_J(S, T) = d_{J(2r+1,r)}(S, T)$.

Proposition 3.10. *In $O(r)$, $d_O(S, T) = 2$ if and only if $d_J(S, T) = 1$. Furthermore, for $\mathcal{D}(O(r) \times P_2) = \begin{bmatrix} D_e & D_o \\ D_o & D_e \end{bmatrix}$, $D_e = 2\mathcal{D}(J(2r + 1, r))$.*

Proof. The first statement follows from equations (7) and (8) (it also follows from the definition). To prove the second part, let $S_0, \ldots, S_{2n}$ be a path of minimum even length between S_0 and S_{2n}, then $S_0, S_2, S_4, \ldots, S_{2n}$ is a path of length n in the Johnson graph by the first statement. Conversely any path of length i in the Johnson graph between S_0 and S_i provides a path of length $2i$ between S_0 and S_i in the odd graph (note that the new vertices are pairwise distinct). This implies the second statement. □

Proposition 3.11. *For $\mathcal{D}(O(r) \times P_2) = \begin{bmatrix} D_e & D_o \\ D_o & D_e \end{bmatrix}$, $D_o = (2r + 1)J - 2\mathcal{D}(J(2r + 1, r))$.*

Proof. Let S and T be two vertices of the odd graph $O(r)$. Let $\widetilde{T}$ be an r-subset of $[2r+1] \setminus T$ containing $S \setminus T$. Thus $S \cap \widetilde{T}$ has the maximum size among all r-subsets of $[2r+1] \setminus T$. Now the minimum odd distance between S and T is one more than the minimum even distance between S and a neighbor of T in the Kneser graph $O(r)$. By Proposition 3.10, this is $1 + 2d_J(S, W) = 1 + 2(r - |S \cap W|)$, where W is a neighbor of T in $O(r)$ that maximizes $|S \cap W|$. It suffices to set $W = \widetilde{T}$. This implies that the minimum odd distance between S and T is

$$\begin{aligned} &1 + 2d_J(S, \widetilde{T}) = 1 + 2(r - |S \cap \widetilde{T}|) = 2r + 1 - 2|S \cap \widetilde{T}| = 2r + 1 - 2(|S \backslash T|) \\ &= 2r + 1 - 2(|S| - |S \cap T|) = 2r + 1 - 2(r - |S \cap T|) = 2r + 1 - 2d_J(S, T). \qquad \square \end{aligned}$$

Theorem 3.12. *Let D be the distance matrix of the Johnson graph $J(2r+1, r)$, let J be the all ones matrix of order $\binom{2r+1}{r}$, and let $r \geq 2$. The distance matrix of the double odd graph $DO(r)$ is*

$$\begin{bmatrix} D_e & D_o \\ D_o & D_e \end{bmatrix} = \begin{bmatrix} 2D & (2r+1)J - 2D \\ (2r+1)J - 2D & 2D \end{bmatrix}$$

and its distance spectrum is

$$\begin{aligned} \operatorname{spec}_{\mathcal{D}}(DO(r)) = \Bigg\{ &(2r+1)\binom{2r+1}{r}, 0^{\left(2\binom{2r+1}{r} - 2r - 2\right)}, \left(-\frac{2s(2r+1, r)}{r}\right)^{(2r)}, \\ &-(2r+1)\binom{2r+1}{r} + 4s(2r+1, r) \Bigg\}. \end{aligned}$$

Proof. The first statement follows from Remark 3.9 and Propositions 3.10 and 3.11. From Equation (6), $\operatorname{spec}(D) = \left\{ s(2r+1, r), 0^{\left(\binom{2r+1}{r} - (2r+1)\right)}, \left(-\frac{s(2r+1,r)}{2r}\right)^{(2r)} \right\}$ where $s(n, r) = \sum_{j=0}^{r} j \binom{r}{j} \binom{n-r}{j}$. Since $J(2r+1, r)$ is distance regular, it is transmission regular, that is, D commutes with J. Then Lemma 3.3 establishes the result. $\square$

We now return to arbitrary Kneser graphs $K(n, r)$ and determine their distance spectra. Let D be the distance matrix of $K(n, r)$. Let A_0 be the identity matrix of order $\binom{n}{r}$ and A_i the adjacency matrix of $J(n; r; i)$ for $1 \leq i \leq r$. It follows that

$$D = \sum_{i=0}^{r} f(i) A_i, \text{ where } f(i) = \min\left\{ 2\left\lceil \frac{i}{n-2r} \right\rceil, 2\left\lceil \frac{r-i}{n-2r} \right\rceil + 1 \right\}.$$

It is known that $\{A_i\}_{i=0}^{r}$ forms an *association scheme* called the *Johnson scheme* and the following properties come from the Corollary to Theorem 2.9 and from Theorem 2.10 in [8]. For more information about association schemes, see, e.g., Section 2.3 in [8].

Theorem 3.13. *(See [8].) The matrices* $\{A_i\}_{i=0}^r$ *form a commuting family and are simultaneously diagonalizable. There are subspaces* $\{V_j\}_{j=0}^r$ *such that*

- *the whole space* $\mathbb{R}^{\binom{n}{r}}$ *is the direct sum of* $\{V_j\}_{j=0}^r$*;*
- *for each* A_i*,*

$$p_i(j) = \sum_{t=0}^{j} (-1)^t \binom{j}{t} \binom{r-j}{i-t} \binom{n-r-j}{i-t}$$

is an eigenvalue with multiplicity $m_j = \frac{n-2j+1}{n-j+1}\binom{n}{j}$ *whose eigenspace is* V_j*.*

The value $p_i(j) = \sum_{t=0}^{j}(-1)^t \binom{j}{t}\binom{r-j}{i-t}\binom{n-r-j}{i-t}$ is known as the *Eberlein polynomial.* Now with Theorem 3.13 the distance spectrum of $K(n,r)$ can also be computed.

Theorem 3.14. *The distance spectrum of* $K(n,r)$ *consists of*

$$\theta_j = \sum_{i=0}^{r} f(i) p_i(j)$$

with multiplicity m_j *for* $j = 0, 1, \ldots, r$*.*

Proof. Since distance matrix of $K(n,r)$ can be written as $\sum_{i=0}^r f(i)A_i$ and $\{A_i\}_{i=0}^r$ forms a commuting family, the distance eigenvalues of $K(n,r)$ are also linear combinations of the eigenvalues of A_i's. Theorem 3.13 gives the common eigenspaces and the eigenvalues with multiplicity, so summing up the corresponding eigenvalues gives the distance eigenvalues of $K(n,r)$. □

Note that Theorem 3.14 gives us the distance spectrum of the odd graph $O(r) = K(2r+1, r)$.

Remark 3.15. The statement in Theorem 3.14 includes all the eigenvalues and multiplicities. However, it does not guarantee $\theta_j \neq \theta_{j'}$ for different j and j'. If this happens, the multiplicity becomes $m_j + m_{j'}$.

4. Number of distinct distance eigenvalues

In this section we construct examples of graphs having few distinct distance eigenvalues and various other properties and establish a lower bound on the number of distinct distance eigenvalues of a tree. Let $q_{\mathcal{D}}(G)$ denote the number of distinct distance eigenvalues of G.

4.1. Fewer distinct distance eigenvalues than diameter

We have answered the following open question:

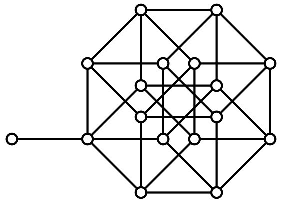

Fig. 2. $Q_4^\ell = Q_4$ with a leaf appended.

Question 4.1. (See [5, Problem 4.3].) Are there connected graphs that are not distance regular with diameter d and having less than $d+1$ distinct distance eigenvalues?

Example 4.2. Consider the graph Q_d^ℓ consisting of Q_d with a leaf appended. Fig. 2 shows Q_4^ℓ. Since Q_d has diameter d, Q_d^ℓ has diameter $d+1$. Since $\mathcal{D}(Q_d)$ has 3 distinct eigenvalues [18, Theorem 2.2] and the eigenvalues of $\mathcal{D}(Q_d^\ell)$ interlace those of $\mathcal{D}(Q_d)$, $\mathcal{D}(Q_d^\ell)$ has at most 5 distinct eigenvalues. For example, $\mathcal{D}(Q_4^\ell)$ has the 5 eigenvalues 0 with multiplicity 11, -8 with multiplicity 3, and the 3 roots of $q(z) = z^3 - 24z^2 - 416z - 640$, which are approximately -10.3149, -1.72176, and 36.0366. However $\operatorname{diam}(Q_4^\ell) + 1 = 6$.

4.2. Few distinct distance eigenvalues and many distinct degrees

The next result follows immediately from Theorem 3.2.

Proposition 4.3. *If G is a transmission regular graph of order n with $\operatorname{spec}_{\mathcal{D}}(G) = \{\rho, \theta_2, \ldots, \theta_n\}$, then*

$$\operatorname{spec}_{\mathcal{D}}(G \,\square\, G) = \{2n\rho\} \cup \{(n\theta_2)^{(2)}, \ldots, (n\theta_n)^{(2)}\} \cup \{0^{((n-1)^2)}\}.$$

Remark 4.4. Let G and H be graphs, and let $\deg(G)$ denote the set of distinct degrees of G. For $x \in V(G)$ and $y \in V(H)$, $\deg_{G \,\square\, H}(x, y) = \deg_G x + \deg_H y$. Thus $\deg(G \,\square\, H) = \deg(G) + \deg(H) := \{a + b : a \in \deg(G), b \in \deg(H)\}$.

There is a graph G with arbitrarily many distinct degrees and $\mathcal{D}(G)$ having exactly 5 distinct eigenvalues.

Example 4.5. Let G be the graph in Fig. 3. Observe that G is transmission regular with distance eigenvalues $\left\{14, \left(\frac{-5+\sqrt{33}}{2}\right)^{(2)}, (-1)^{(4)}, \left(\frac{-5-\sqrt{33}}{2}\right)^{(2)}\right\}$. Furthermore, $\deg(G) = \{3, 4\}$. Let G^i be $G \,\square \cdots \square\, G$ with i copies of G. By Proposition 4.3, $q_{\mathcal{D}}(G \,\square\, G) = 5$ and one of the distance eigenvalues is zero, and by induction $q_{\mathcal{D}}(G^{2^k}) = 5$. By inductively applying Remark 4.4, G^{2^k} has at least $2^k + 1$ distinct degrees.

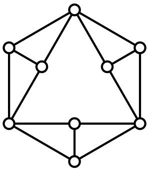

Fig. 3. Transmission regular graph G with 4 eigenvalues that is not regular.

4.3. Minimum number of distinct distance eigenvalues of a tree

In studies of matrices or families of matrices associated with a graph G, it is sometimes the case that $\operatorname{diam}(G) + 1$ is a lower bound for the number of distinct eigenvalues for some or all graphs G. In many situations a real symmetric $n \times n$ matrix A is studied by means of its graph $\mathcal{G}(A)$, which has vertices $\{1, \ldots, n\}$ and edges ij exactly where $a_{ij} \neq 0$. It is well known that a nonnegative matrix A has at least $\operatorname{diam}(\mathcal{G}(A)) + 1$ distinct eigenvalues [11, Theorem 2.2.1], and any real symmetric matrix A has at least $\operatorname{diam}(\mathcal{G}(A))+1$ distinct eigenvalues if $\mathcal{G}(A)$ is a tree [21, Theorem 2].[3] Note that $\mathcal{G}(\mathcal{D}(T))$ is not a tree even when T is a tree.

Question 4.6. For a tree T, is the number of distinct distance eigenvalues $q_{\mathcal{D}}(T)$ at least $\operatorname{diam}(T) + 1$?

The answer is yes for all trees of order at most 20, as determined through computations in *Sage* [22]. We can prove the following weaker bound.

Proposition 4.7. *Let T be a tree. Then T has at least $\lceil \frac{\operatorname{diam}(T)}{2} \rceil$ distinct distance eigenvalues.*

Proof. Let $L(T)$ be the line graph of T and D be the distance matrix of T. Then $\operatorname{diam}(L(T)) = \operatorname{diam}(T) - 1$. By [11, Theorem 2.2.1], the adjacency matrix $\mathcal{A}(L(T))$ of $L(T)$ has at least $\operatorname{diam}(L(T)) + 1 = \operatorname{diam}(T)$ distinct eigenvalues. Let $K = 2I + \mathcal{A}(L(T))$. By [23], the eigenvalues of $-2K^{-1}$ interlace the eigenvalues of D (with D having one more). Since $-2K^{-1}$ also has at least $\operatorname{diam}(T)$ distinct eigenvalues, D has at least $\lceil \frac{\operatorname{diam}(T)}{2} \rceil$ distinct eigenvalues. $\square$

4.4. Zero forcing bound for the number of distinct distance eigenvalues

Let G be a graph. Then $\mathcal{S}(G)$ denotes the family of real symmetric matrices A such that $\mathcal{G}(A) = G$. The *maximum nullity* of G is defined to be the largest possible nullity among matrices in $\mathcal{S}(G)$, and is denoted by $\mathrm{M}(G)$. The *zero forcing number* of a graph G, $\mathrm{Z}(G)$, is the minimum cardinality of a set S of blue vertices (the rest of the

[3] In [21], $d(G) = \operatorname{diam}(G) + 1$.

vertices are white) such that all the vertices are turned blue after iteratively applying the "color-change rule": a white vertex is turned blue if it is the only white neighbor of a blue vertex. Zero forcing was introduced in [2]. The zero forcing number $\mathrm{Z}(G)$ is used to bound the maximum nullity:

Theorem 4.8. *(See [2].) For any graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.*

Theorem 4.9. *For any graph G of order n, the number of distinct eigenvalues of $\mathcal{D}(G)$ is at least*

$$\frac{n-1}{\mathrm{Z}(\overline{G})+1}+1.$$

Proof. Let θ be an eigenvalue of $\mathcal{D}(G)$. Then the multiplicity m_θ is $\mathrm{null}(\mathcal{D}(G)-\theta I)$. Now observe that $\mathcal{D}(G)-J-\theta I$ is a matrix in $\mathcal{S}(\overline{G})$, so $\mathrm{null}(\mathcal{D}(G)-J-\theta I) \leq \mathrm{M}(\overline{G}) \leq \mathrm{Z}(\overline{G})$ by Theorem 4.8. Since J is a rank-1 matrix,

$$m_\theta = \mathrm{null}(\mathcal{D}(G)-\theta I) \leq \mathrm{null}(\mathcal{D}(G)-J-\theta I)+1 \leq \mathrm{Z}(\overline{G})+1.$$

Each eigenvalue has multiplicity at most $\mathrm{Z}(\overline{G})+1$. Also, since $\mathcal{D}(G)$ is a irreducible matrix, by the Perron–Frobenius Theorem, the largest eigenvalue has multiplicity one. As a consequence, the number of distinct eigenvalues is at least $\frac{n-1}{\mathrm{Z}(\overline{G})+1}+1$. $\square$

The bound given in Theorem 4.9 can be tight, as seen in the next example.

Example 4.10. All Hamming graphs $H(d,n)$ on $d \geq 3$ dimensions (including $Q_d := H(d,2)$) have 3 distance eigenvalues [18]. Assume $d \geq 3$ and in Q_d let $a := 1000\cdots0$, $b := 0100\cdots0$, $c := 0010\cdots0$, $e := 1100\cdots0$, $p := 0000\cdots0$, and $q := 0110\cdots0$. Observe that $V(Q_d)\setminus\{a,b,c,e\}$ is a zero forcing set for $\overline{Q_d}$ using the following forces: $p \to e$, $e \to c$, $q \to a$, and $c \to b$. Therefore, $\mathrm{Z}(\overline{Q_d}) \leq |\overline{Q_d}|-4$ and $\left\lceil \frac{|Q_d|-1}{\mathrm{Z}(\overline{Q_d})+1}+1 \right\rceil = 3$. Thus the bound given in Theorem 4.9 is tight for hypercubes Q_d with $d \geq 3$; more generally, the bound is tight for $H(d,n)$ for $d \geq 3$.

5. Determinants and inertias of distance matrices of barbells and lollipops

A *lollipop graph*, denoted by $L(k,\ell)$ for $k \geq 2$ and $\ell \geq 0$, is constructed by attaching a k-clique by an edge to one pendant vertex of a path on ℓ vertices.[4] A *barbell graph*, denoted by $B(k,\ell)$ for $k \geq 2$ and $\ell \geq 0$, is a graph constructed by attaching a k-clique by an edge to each of the two pendant vertices of a path on ℓ vertices. Define the family of *generalized barbell graphs*, denoted by $B(k;m;\ell)$ for $k,m \geq 2$ and $\ell \geq 0$, to be the

[4] In the literature the name 'lollipop' is often used to denote a graph obtained by appending a cycle rather than a clique to the pendant vertex of the path.

graph constructed by attaching a k-clique by an edge to one end vertex of a path on ℓ vertices, and attaching an m-clique by an edge to the other end vertex of the path. Thus $B(k,\ell) = B(k;k;\ell)$ for $k \geq 2$ and $\ell \geq 0$, and $L(k,\ell) = B(k;2;\ell-2)$ for $\ell \geq 2$. In this section we establish the determinant and inertia of the distance matrices of $B(k;m;\ell)$, and hence for barbells and lollipops.

The technique we use is the method of quotient matrices, which has been applied to distance matrices previously (see, for example, [5]). Let A be a block matrix whose rows and columns are partitioned according to some partition $X = \{X_1, \dots, X_m\}$ with *characteristic matrix* S, i.e., i,j-entry of S is 1 if $i \in X_j$ and 0 otherwise. The *quotient matrix* $B = [b_{ij}]$ of A for this partition is the $m \times m$ matrix whose entries are the average row sums of the blocks of A, i.e. b_{ij} is the average row sum of the block A_{ij} of A for each i,j. The partition X is called *equitable* if the row sum of each block A_{ij} is constant, that is $AS = SB$. It is well known that if the partition is equitable, then the spectrum of A consists of the spectrum of the quotient matrix B together with the eigenvalues belonging to eigenvectors orthogonal to the columns of S (see, e.g., [10, Lemma 2.3.1]).

Theorem 5.1. *The distance determinants of generalized barbell graphs are given by*

$$\det \mathcal{D}(B(k;m;\ell)) = (-1)^{k+m+\ell-1} 2^{\ell}(km(\ell+5) - 2(k+m)).$$

Proof. We label the vertices of $B(k;m;\ell)$ as follows: $v_1, \dots, v_k$ and $v_{k+1}, \dots, v_{k+m}$ are the cliques on k and m vertices, respectively; $v_{k+m+1}, \dots, v_{k+m+\ell}$ are the vertices on the path with v_i adjacent to v_{i+1}; finally v_k is adjacent to v_{k+m+1} and v_{k+m} is adjacent to $v_{k+m+\ell}$.

Let $\mathcal{Q}(B(k;m;\ell))$ be the quotient matrix of the distance matrix of $B(k;m;\ell)$, partitioning the rows and columns of $\mathcal{D}(B(k;m;\ell))$ according to $X = \{X_1, X_2, \dots, X_{\ell+4}\}$ where $X_1 = \{v_1, \dots, v_{k-1}\}$, $X_2 = \{v_{k+1}, \dots, v_{k+m-1}\}$, $X_3 = \{v_k\}$, and $X_i = \{v_{k+m+i-4}\}$ for $i = 4, \dots, \ell+4$. Clearly, $\mathcal{D}(B(k;m;\ell))$ has $(k-2)+(m-2)$ eigenvectors for eigenvalue -1 that are orthogonal to the columns of the characteristic matrix S, so $\det \mathcal{D}(B(k;m;\ell)) = (-1)^{(k-2)+(m-2)} \det \mathcal{Q}(B(k;m;\ell)) = (-1)^{k+m} \det \mathcal{Q}(B(k;m;\ell))$. Thus it suffices to compute the determinant of the $(\ell+4) \times (\ell+4)$ quotient matrix. For $\ell = 0$, simply compute the determinant of the 4×4 quotient matrix (the upper left block of the block matrix in (9) below). Now assume $\ell \geq 1$.

$$\det \mathcal{Q}(B(k;m;\ell)) = \det \left[\begin{array}{cccc|cccc} k-2 & (\ell+3)(m-1) & 1 & \ell+2 & 2 & 3 & \cdots & \ell+1 \\ (\ell+3)(k-1) & m-2 & \ell+2 & 1 & \ell+1 & \ell & \cdots & 2 \\ k-1 & (\ell+2)(m-1) & 0 & \ell+1 & 1 & 2 & \cdots & \ell \\ (\ell+2)(k-1) & m-1 & \ell+1 & 0 & \ell & \ell-1 & \cdots & 1 \\ \hline 2(k-1) & (\ell+1)(m-1) & 1 & \ell & 0 & 1 & \cdots & \ell-1 \\ 3(k-1) & \ell(m-1) & 2 & \ell-1 & 1 & 0 & \cdots & \ell-2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\ell+1)(k-1) & 2(m-1) & \ell & 1 & \ell-1 & \ell-2 & \cdots & 0 \end{array}\right] \tag{9}$$

$$= \det \left[\begin{array}{cccc|ccccc} k-2 & (\ell+3)(m-1) & 1 & \ell+2 & 2 & 3 & \cdots & \ell & \ell+1 \\ (\ell+3)(k-1) & m-2 & \ell+2 & 1 & \ell+1 & \ell & \cdots & 3 & 2 \\ 1 & -(m-1) & -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -(k-1) & 1 & -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ \hline k-1 & -(m-1) & 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ k-1 & -(m-1) & 1 & -1 & 1 & -1 & \cdots & -1 & -1 \\ k-1 & -(m-1) & 1 & -1 & 1 & 1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k-1 & -(m-1) & 1 & -1 & 1 & 1 & \cdots & 1 & -1 \end{array}\right] \tag{10}$$

$$= \det \left[\begin{array}{cccc|cccc} k-2 & (\ell+3)(m-1) & 1 & \ell+2 & 2 & 3 & \cdots & \ell+1 \\ (\ell+3)(k-1) & m-2 & \ell+2 & 1 & \ell+1 & \ell & \cdots & 2 \\ 1 & -(m-1) & -1 & -1 & -1 & -1 & \cdots & -1 \\ -k & m & 0 & 0 & 0 & 0 & \cdots & 0 \\ 2(k-1) & -m & 2 & 0 & 0 & 0 & \cdots & 0 \\ \hline & & & & & & & 0 \\ & 0_{(\ell-1)\times 4} & & & & 2I_{\ell-1} & & \vdots \\ & & & & & & & 0 \end{array}\right] \tag{11}$$

$$= (-2)^{\ell-1} \det \begin{bmatrix} k-2 & (\ell+3)(m-1) & 1 & \ell+2 & \ell+1 \\ (\ell+3)(k-1) & m-2 & \ell+2 & 1 & 2 \\ 1 & -(m-1) & -1 & -1 & -1 \\ -k & m & 0 & 0 & 0 \\ 2(k-1) & -m & 2 & 0 & 0 \end{bmatrix}$$

$$= (-1)^{\ell-1}2^{\ell-1}2(km(\ell+5) - 2(k+m)).$$

The matrix (10) is obtained from (9) by iteratively subtracting the ith row from $(i+1)$th row starting from $i = \ell+3$ to $i = 5$ and subtracting ith row from $(i+2)$th row for $i = 3, 2, 1$. The matrix (11) is obtained from (10) by iteratively subtracting the jth row from $(j+1)$th starting from $j = \ell+3$ to $j = 3$. □

Theorem 5.2. *The inertia of $\mathcal{D}(B(k; m; \ell))$ is $(n_+, n_0, n_-) = (1, 0, k+m+\ell-1)$.*

Proof. We use induction on $k+m$. The base case is $k = m = 2$, which is a path. Since the inertia of any tree on n vertices is $(1, 0, n-1)$ [17], the inertia of $\mathcal{D}(B(2; 2; \ell))$ is $(1, 0, \ell+3)$. Thus the assertion follows for $k+m = 4$. Let $\theta_1 \geq \cdots \geq \theta_{k+m+\ell+1}$ be the eigenvalues of $\mathcal{D}(B(k+1; m; \ell))$ and $\mu_1 \geq \cdots \geq \mu_{k+m+\ell}$ be the eigenvalues of $\mathcal{D}(B(k; m; \ell))$. By interlacing we have $\mu_i \geq \theta_{i+1}$, for $i = 1, \ldots, k+m+\ell$. The induction hypothesis implies that $\theta_3 \geq \cdots \geq \theta_{k+m+\ell}$ are negative numbers and $\theta_1 > 0$. By Theorem 5.1, the determinant will change sign, so we obtain $\theta_2 < 0$, completing the proof. □

Corollary 5.3. *The distance determinants of barbell graphs are given by*

$$\det \mathcal{D}(B(k,\ell)) = (-1)^{\ell-1}2^{\ell}(k^2(\ell+5)-4k).$$

Corollary 5.4. *The distance determinants of lollipop graphs are given by*

$$\det \mathcal{D}(L(k,\ell)) = (-1)^{k+\ell-1}2^{\ell-1}\left(k(\ell+2)-2\right).$$

Proof. The case $\ell \geq 2$ follows from Theorem 5.1 because $L(k,\ell) = B(k;2;\ell-2)$. The proof for $\ell = 1$ is a simplified version of the proof of that theorem. The eigenvalues of the k-clique $L(k,0)$ are $\{k-1, (-1)^{(k-1)}\}$, so $\det \mathcal{D}(L(k,0)) = (-1)^{k-1}(k-1)$, which agrees with $(-1)^{k+\ell-1}2^{\ell-1}\left(k(\ell+2)-2\right)$ for $\ell = 0$. $\square$

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