Linear Algebra and its Applications 446 (2014) 192–195

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Counterexamples to an edge spread question for zero forcing number

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A R T I C L E I N F O

Article history:
Received 24 August 2013
Accepted 9 January 2014
Available online 23 January 2014
Submitted by R. Brualdi

MSC:
05C50
15A03
15B57

Keywords:
Minimum rank
Maximal nullity
Zero forcing number
Edge spread

A B S T R A C T

This note gives counterexamples to an edge spread problem on the zero forcing number.

1. Introduction

For a simple graph G on n vertices, the *minimum rank problem* is to determine the *minimum rank* $\mathrm{mr}(G)$ of G, which is the smallest possible rank among all $n \times n$ symmetric

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[1] This research is partially supported by the National Science Council of the Republic of China under grant NSC101-2115-M-002-005-MY3.

http://dx.doi.org/10.1016/j.laa.2014.01.009

matrices whose off-diagonal ij-entry is nonzero whenever ij is an edge in G and zero otherwise. Equivalently, we may consider the *maximal nullity* $M(G) := n - \mathrm{mr}(G)$. In 2008, a graph parameter called the *zero forcing number* was introduced by the AIM Minimum Rank-Special Graphs Work Group [1] to serve as an upper bound for the maximal nullity.

Let G be a graph with each vertex colored black or white. A *zero forcing process* on G is defined by the *color-change rule*: a black vertex x can force a white vertex y to black at the next step when y is the only white neighbor of x. A *zero forcing set* of G is a subset $B \subseteq V(G)$ which as the initial set of black vertices can force all vertices in $V(G)$ to black. The *zero forcing number* $Z(G)$ is the minimum size of a zero forcing set of G. It was proved in [1] that

$$M(G) \leqslant Z(G) \quad \text{for any graph } G.$$

To describe a zero forcing process, we write $x \to y$ to denote that at some stage a black vertex x forces its only white neighbor y to black. The *chronological list* of a zero forcing process is a record $(x_i \to y_i\colon\ 1 \leqslant i \leqslant h)$ in order. Thus when we mention a zero forcing process, we mean the corresponding zero forcing set together with its chronological list.

A *maximal chain* is a maximal sequence of vertices of the form

$$x_1 \to x_2 \to \cdots \to x_k.$$

It is worth noting that every maximal chain is an induced path in G, by the color-change rule. Also, the set of all maximal chains of a zero forcing process is a collection of induced paths that cover all vertices of the graph.

The *edge spread* $z_e(G)$ of a graph G on an edge e is

$$z_e(G) = Z(G) - Z(G - e).$$

Edholm et al. [2] proved that if $z_e(G) = -1$, then the edge e is an edge in some maximal chain of every optimal zero forcing process. Here an optimal zero forcing process means one whose corresponding zero forcing set is of size $Z(G)$. In the same paper, the authors asked in Question 2.22 if the converse of the previous statement is also true. The purpose of this note is to give a negative answer to the question by constructing infinitely many counterexamples.

2. The robot graphs

The counterexamples provided in this note are the *robot graphs* $R_n(h_1, h_2, \ldots, h_{2n})$ to be constructed as follows. Choose integers $n \geqslant 2$, $h_0 = 2$ and $h_s \geqslant 3$ for $1 \leqslant s \leqslant 2n$. Let P_s be the path with the h_s vertices labeled $p_{s,1}, p_{s,2}, \ldots, p_{s,h_s}$ for $0 \leqslant s \leqslant 2n$; and let C_{2n+1} be the cycle with the $2n+1$ vertices labeled $v_0, v_1, v_2, \ldots, v_{2n}$. Define H_p to be the graph obtained from $C_{2n+1}, P_0, P_1, \ldots, P_{2n}$ by identifying p_{s,h_s} with v_s for $0 \leqslant s \leqslant 2n$,

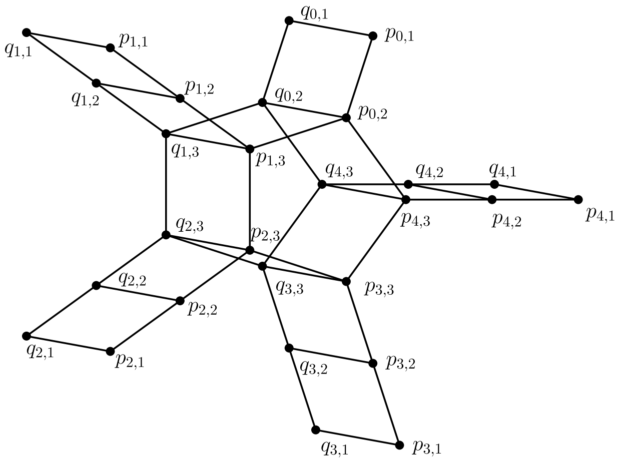

Fig. 1. Robot graph $R_2(3,3,3,3)$.

adopting the name p_{s,h_s}. Next, make a copy H_q of H_p and label the vertex corresponding to $p_{s,t}$ as $q_{s,t}$. The *robot graph* $R_n(h_1,h_2,\dots,h_{2n})$ is defined as the graph obtained from the disjoint union of H_p and H_q by adding edges $p_{s,t}q_{s,t}$ for $0 \leqslant s \leqslant 2n$ and $1 \leqslant t \leqslant h_s$; see Fig. 1 for an example. This graph is the Cartesian product of H_p with a path on 2 vertices.

Theorem 1. *If $G = R_n(h_1,h_2,\dots,h_{2n})$ is a robot graph in which $e = p_{0,1}q_{0,1}$, then $Z(G) = Z(G-e) = 2n+1$ and e is in some maximal chain of every optimal zero forcing process ψ of G.*

Proof. We first prove that $Z(G) = 2n+1$. Notice that $\{p_{0,1}, p_{1,1}, q_{1,1}\} \cup \{p_{2s,1}, q_{2s,1}\colon 1 \leqslant s \leqslant n-1\}$ is a zero forcing set of size $2n+1$, which gives that $Z(G) \leqslant 2n+1$. On the other hand, to see $Z(G) \geqslant 2n+1$ we only need to prove that ψ has at least $2n+1$ maximal chains. We claim that for $0 \leqslant s \leqslant 2n$, either $p_{s,1}$ or $q_{s,1}$ is an end vertex of some maximal chain of ψ. For otherwise, $p_{s,1}$ and $q_{s,1}$ being middle vertices of some maximal chain would imply that $p_{s,2}, p_{s,1}, q_{s,1}, q_{s,2}$ are in the same maximal chain, violating that the chain is an induced path.

For $0 \leqslant s \leqslant 2n$, define $L_s := \{p_{s,t}, q_{s,t}\colon 1 \leqslant t \leqslant h_s\}$. Then we claim that for $1 \leqslant s \leqslant 2n$, L_s contains two end vertices of different maximal chains of ψ. If $p_{s,1}$ and $q_{s,1}$ are in distinct maximal chains, the claim holds, so assume they are in the same maximal chain. There are two possibilities: either $p_{s,1}$ and $q_{s,1}$ are end vertices of the same maximal chain of ψ or one of $p_{s,1}$ and $q_{s,1}$, say $q_{s,1}$, is a middle point of some maximal chain of ψ. For the former case, $p_{s,1}$ and $q_{s,1}$ form a maximal chain of ψ of length 2. Then, either $p_{s,2}$ or $q_{s,2}$ is an end vertex of another maximal chain of ψ. For the latter case, $p_{s,1}, q_{s,1}, q_{s,2}$ are in the same maximal chain and so $p_{s,2}$ is in another maximal chain μ. As $p_{s,2}$ is of degree 3 in G, it is adjacent to just one vertex in μ and so is an end vertex of μ. Hence the claim holds.

Therefore, there are a total of at least $4n+1$ end vertices. Since each maximal chain contains only two end vertices, at least $2n+1 = \lceil \frac{4n+1}{2} \rceil$ maximal chains are required. This ensures that $Z(G) = 2n+1$.

Next, we show that $e = p_{0,1}q_{0,1}$ is in some maximal chain of ψ. Suppose to the contrary that e is not in any maximal chain of ψ. Since e is not used in any maximal chain, both $p_{0,1}$ and $q_{0,1}$ are end vertices of different maximal chains of ψ. This together with the optimality of ψ and the arguments in the previous paragraph gives exactly $4n+2$ end vertices of maximal chains of ψ. Also, every maximal chain μ of ψ has one end vertex in L_s and the other end vertex in L_t for $0 \leqslant s < t \leqslant 2n$. Then μ contains at least two vertices in the set $W = \{p_{s,h_s}, q_{s,h_s}\colon 0 \leqslant s \leqslant 2n\}$. Since $2n+1$ maximal chains are needed and $|W| = 4n+2$, each maximal chain contains exactly two vertices in W. Furthermore, the two vertices of each chain should be adjacent, and so the pair must be of the form of $\{p_{s,h_s}, p_{t,h_t}\}$ or $\{q_{s,h_s}, q_{t,h_t}\}$. But there are at most n disjoint sets of the form $\{p_{s,h_s}, p_{t,h_t}\}$ and at most n disjoint sets of the form $\{q_{s,h_s}, q_{t,h_t}\}$, violating that there are $2n+1$ maximal chains of ψ. Therefore e is in some maximal chain of every optimal ψ.

The remaining task is to prove that $Z(G-e) = 2n+1$. Since $\{p_{0,1}\} \cup \{p_{2s+1,1}, q_{2s+1,1}\colon 0 \leqslant s \leqslant n-1\}$ is a zero forcing set of $G-e$, we have $Z(G-e) \leqslant 2n+1$. On the other hand, precisely the same argument as for G gives that there are at least $4n+2$ end vertices of maximal chains of every optimal zero forcing process of $G-e$. Again, $Z(G-e) \geqslant \frac{4n+2}{2} = 2n+1$ and so $Z(G-e) = 2n+1$. $\square$

Example 2. The graph in Fig. 1 is a robot graph with $n=2$ and $h_1 = h_2 = h_3 = h_4 = 3$. It has the zero forcing number 5, and every optimal zero forcing process uses the edge $p_{0,1}q_{0,1}$ to perform a force. Also, the zero forcing number remains 5 when the edge $p_{0,1}q_{0,1}$ is deleted.

It is notable that the condition $n \geqslant 2$ is necessary, since the zero forcing number of $R_1(3,3) - p_{0,1}q_{0,1}$ is $4 > 3 = 2n+1$.

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