6.1 Inverse functions

One-to-one functions

A function f is called a *one-to-one function* if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

How to determine whether a function is one-to-one?

- By definition.
  Example:

- Horizontal line test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.
  Example:

- Strictly increasing and strictly decreasing functions are one-to-one.
  Example:

Problem 1 *Is $f(x) = x + 1$ a one-to-one function?*

Multiple Choice:

(a) *Incorrect*

(b) *Not sure*

(c) *Click here?* ✓

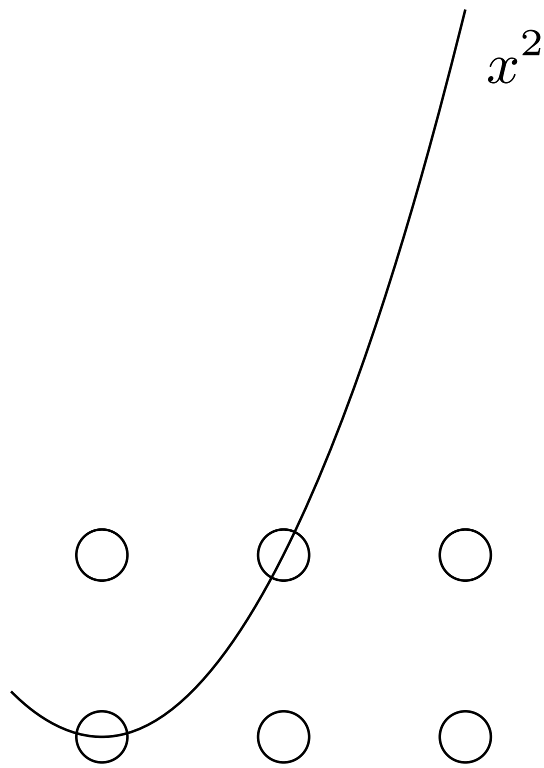
(d) *Not me!*

Graph of $a = 1, ax^2$

$3 \times 2 = \boxed{6}$

Question 2 $\frac{\partial}{\partial x} x^3 \sin(y) = \boxed{3x^2 sin(y)}$

Question 3 ***Hint:*** *3×2 is the number of objects in 3 groups of 2 objects*



Hint:

Hint: $3 \times 2 = 6$

$3 \times 2 = \boxed{6}$

Something unimportant something important somthing unimportant something really important.

Question 4 *What is the abscissa of the critical point of the function $f(x) = x^2 + 2x + 1$?*

Hint: *What is the derivative of f?*

$f(x) = \boxed{2x+2}$

$x = \boxed{-1}$

Inverse function

Let f be a one-to-one function with domain A and range B. Then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \tag{1}$$

for any y in B.

Caution: Do not mistake the -1 in f^{-1} for an exponent. $f^{-1}(x)$ doen not mean $\frac{1}{f(x)}$!!

Properties:

- domain of f^{-1} = range of f; range of f^{-1} = domain of f.
- $f^{-1}(x) = y \Leftrightarrow f(y) = x$
- $f^{-1}(f(x)) = x$ for every x in A; $f(f^{-1}(y)) = y$ for every y in B
- The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example:

How to find the inverse function of a one-to-one function?

(a) Write $y = f(x)$.

(b) Solve this equation for x in terms of y (if possible).

(c) To express f^{-1} as a function of x, interchange x and y.

The resulting equation is $y = f^{-1}(x)$.

Example:

The Calculus of Inverse Functions

Recall that a function f is continuous if its graph has no break (consists of just one piece).

Theotem: If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Theorem: If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiabla at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Example:

6.2* The Natural Logarithmic Function

The *natural logarithmic function* is the function defined by

$$\ln x = \int_1^x \frac{1}{t} dt \qquad x > 0$$

Remark: e is the number such that ln $e = 1$ ($e \approx 2.718$).

Properties:

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- (Laws of Logarithms) If x and y are positive numbers and r is a rational number, then

  (a) $\ln(xy) = \ln x + \ln y$

  (b) $\ln(\frac{x}{y}) = \ln x - \ln y$

  (c) $\ln(x^r) = r \ln x$

  Example:

- (Limits) $\lim_{x\to\infty} \ln x = \infty$; $\lim_{x\to 0^+} \ln x = -\infty$

  Graph of $y = \ln x$

  Example:

- $\dfrac{d}{dx}(\ln\, u) = \dfrac{1}{u}\,\dfrac{du}{dx}$; or $\dfrac{d}{dx}[\ln\, g(x)] = \dfrac{g'(x)}{g(x)}$
  Example:

- (Some important formulas)

  (a) $\dfrac{d}{dx}(\ln\,|x|) = \dfrac{1}{x}$

  (b) $\displaystyle\int \frac{1}{x}\, dx = \ln\,|x| + C$

  (c) $\displaystyle\int \tan\, x\, dx = \ln\,|\sec\, x| + C$

  Example:

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms.

Steps in Logarithmic Differentiation

(a) Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.

(b) Differentiate implicitly with respect to x.

(c) Solve the resulting equation for y'.

Example: