Mathematical Techniques for Physicists

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Preface

This book is written for an undergraduate Mathematical Methods course, designed for physics students. A knowledge of single and multivariable calculus is required for a complete understanding of the material covered. Since the range of topics presented is wide, it is not our goal to treat each with much rigor but instead to provide a more methodical view that will serve as an introduction and expose the reader to the topics that will appear again in higher level physics course:

1. Fourier Analysis,
2. Vector Calculus,
3. Linear Algebra, and
4. Partial Differential Equations.

Each section is kept relatively brief in order to allow the reader to more deeply connect with the key concepts and ideas presented and finishes with a set of reading questions and/or in-class exercises. It is recommended to complete the reading questions after reading the section to test your understanding. The in-class exercises can be saved for designated class time.

This book also includes a computational component using Python 3.7. Throughout, there are exercises that use Python. However, we do not provide an introduction to the Python language. See Chapter 5 for resources and information on installing and using the Anaconda Python distribution.

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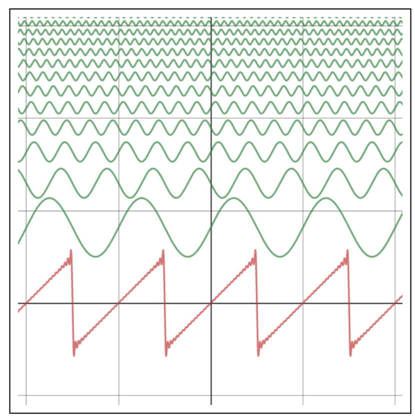
Chapter 1

Fourier Analysis

Fourier analysis is the study of the way that functions can be approximated as linear combinations of trigonometric functions. Similar to a Taylor expansion, the approximation becomes exact with an infinite number of trigonometric terms added.



Due to its prevalence in partial differential equations, number theory, combinatorics, signal processing, digital image processing, probability theory, statistics, forensics, option pricing, cryptography, numerical analysis, acoustics, oceanography, sonar, optics, diffraction, geometry, protein structure analysis, and many more fields, Fourier analysis is crucial to the developing physicist's education.

We will begin by discussing notation and convention of complex numbers. Then, we will explore the algorithm for expanding functions as a Fourier and complex Fourier series, as well the Fourier and Laplace integral transforms.

In this chapter you will become comfortable with operations and notation for complex numbers, develop an introductory proficiency for operating in the frequency domain, and gain computational competence while working with integral transforms and the Dirac Delta function.

1.1 Complex Numbers

Real numbers include rational and irrational numbers. It is impossible to list all real numbers or even begin such a list, but some examples are 3, π, -2^{274}, 0, $\frac{-3}{20}$, and 0.040201. A **complex number** is of the form $a + bi$, where a and b are real numbers and i is the complex unit satisfying $i^2 = -1$.

Similar to the way that real numbers can be visualized on a line, an **Argand diagram** (also called a **Gauss Plane**) can be used to visualize complex numbers since the complex numbers are similar to ordered pairs of real numbers. We can plot an arbitrary complex number $z = x + iy$ on the Argand diagram below.

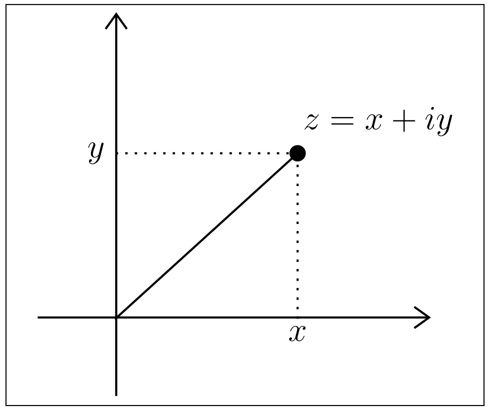


Figure 1.1: Argand Diagram

The horizontal axis is called the real axis and the vertical axis is called the imaginary axis. For an arbitrary complex number $z = x + iy$, we define

- $\mathrm{Re}(z) = x$, "the real part of z," and
- $\mathrm{Im}(z) = y$, "the imaginary part of z." (Remember that $\mathrm{Im}(z)$ is a real number!)

The complex numbers have the following operations defined. For arbitrary $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, and scalar (real) c,

- addition and subtraction
$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2), \tag{1.1}$$
- scalar multiplication
$$cz_1 = (cx_1) + i(cy_2), \tag{1.2}$$
- multiplication
$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), \tag{1.3}$$
- division
$$\frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{|z_2|^2} = \frac{x_1x_2 + y_2y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{1.4}$$

The **complex conjugate** of a complex number $z = x + iy$ is the complex number $z^* = x - iy$, whose imaginary part differs in sign from z. The **modulus** of a complex number z is written $|z|$ and is defined as

$$|z| = \sqrt{\mathrm{Re}^2(x) + \mathrm{Im}^2(x)}. \tag{1.5}$$

For an arbitrary $z = x + iy$, it is always true that

$$z \cdot z^* = z^* \cdot z = |z|^2 = x^2 + y^2. \tag{1.6}$$

We can also write complex numbers using a polar representation.

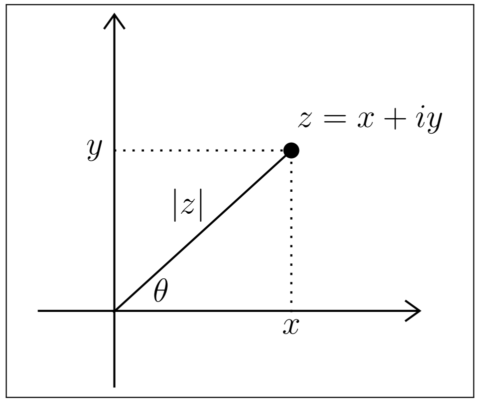


Figure 1.2: Polar Representation of Complex Numbers

Write θ for the angle between the complex 'vector', z, and the real axis and write $r = \sqrt{x^2 + y^2} = |z|$. Then we have that $x = r\cos\theta$ and $y = r\sin\theta$. Recall Euler's formula,

$$e^{ix} = \cos x + i\sin x. \tag{1.7}$$

We can use these two facts to write

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}. \tag{1.8}$$

Reading Questions

Reading Question 1.1.1. Explain why all real numbers are complex numbers.

Reading Question 1.1.2. What is i^n for all $n = 0, 1, 2, 3, 4, \ldots$?

Reading Question 1.1.3. What is $(z^*)^*$ for any complex z? What is z^* for any real z?

Reading Question 1.1.4. On an Argand diagram, how can taking the complex conjugate of a complex number be visualized? How about multiplying a complex number by i.

Reading Question 1.1.5. For each of the operations defined for complex numbers that are listed above, make sure that they follow intuitively. Perhaps, treat i as a 'real variable' and 'derive' each from the same operations defined on real numbers.

Reading Question 1.1.6. Prove Eq. 1.6.

Reading Question 1.1.7. Prove Euler's formula (Eq. 1.7) using a Taylor expansion.

Reading Question 1.1.8. For an arbitrary $z = x + iy$, when is $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and when is it not?

In-Class Exercises

In-Class Exercise 1.1.9. Two complex numbers z and w are given by $z = 3 + 4i$ and $w = 2 - i$. On an Argand diagram, plot (a) $z + w$, (b) $w - z$, (c) wz, (d) z/w, (e) $z^*w + w^*z$, and (f) w^2. (Exercise 3.1 from [1].)

In-Class Exercise 1.1.10. Find the polar representation of the following complex numbers:

(a) $-\sqrt{3} + i$

(b) $2 - 2i$

(c) $-i3/2$

(d) $(-1 - \sqrt{3}i)/2$

(e) i

(f) -2

In-Class Exercise 1.1.11. Does $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ hold for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$? Check that with $z_1 = \sqrt{3} + i$ and $z_2 = (3/2)(1 + i\sqrt{3})$.

In-Class Exercise 1.1.12. By considering the real and imaginary parts of the product $e^{i\theta}e^{i\phi}$ prove the standard formulae for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$. (Exercise 3.2 from [1].)

1.2 Fourier Series

While more in-depth analysis of functions requires a more rigorous definition, we will be sufficient to consider a **function** a rule f, that takes a variable input value x and outputs a value $f(x)$. Do not confuse functions with their output values! A common bad habit is to refer to "the function $f(x)$" but remember: $f(x)$ is not a function, rather, it is the value of the function f evaluated at x. We say that a real-valued function is **periodic** if there is some real $L > 0$ for which every real x satisfies $f(x) = f(x + L)$.

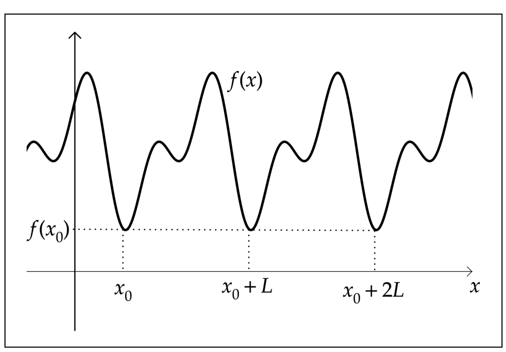


Figure 1.3: An Arbitrary Periodic Function

Consider a periodic function f of a real variable x. Let x_0 be an arbitrary position on the x (real, horizontal) axis and let L be the period of f. It turns out that any such periodic function can be expanded in Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi}{L} rx\right) + b_r \sin\left(\frac{2\pi}{L} rx\right) \right], \tag{1.9}$$

with **Fourier coefficients**

$$a_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi}{L} rx\right) \, dx, \text{ and} \tag{1.10}$$

$$b_r = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi}{L} rx\right) \, dx. \tag{1.11}$$

Note that for a_r, $r = 0, 1, 2, 3, \ldots$ and for b_r, $r = 1, 2, 3, \ldots$. Each term of the infinite sum is called a **Fourier component**. When computing these values, x_0 can be chosen to make the integral easiest.

The **Theorem of Discontinuity** states that at discontinuous points, x_d, the Fourier series converges to

$$f(x_d) = \frac{1}{2} \left(\lim_{x \to x_d^+} f(x) + \lim_{x \to x_d^-} f(x) \right). \tag{1.12}$$

This means that given any periodic function, we can write this function as the infinite linear combination of sin and cos waves of different frequencies.

Example 1.2.1. Find the Fourier series for the Saw-tooth Wave defined by $f(x) = x$ for $\frac{-L}{2} < x < \frac{L}{2}$ with period L.

Choose $x_0 = -L/2$ and evaluate

$$a_r = \frac{2}{L} \int_{-L/2}^{L/2} x \cos\left(\frac{2\pi}{L} rx\right) \, dx = 0 \text{ and}$$

$$b_r = \frac{2}{L} \int_{-L/2}^{L/2} x \sin\left(\frac{2\pi}{L} rx\right) \, dx = \frac{-L}{\pi r} \cos(\pi r) = \frac{L}{\pi r} (-1)^{r+1}.$$

Then,

$$f(x) = \sum_{r=1}^{\infty} \left[\frac{L}{\pi r} (-1)^{r+1} \sin\left(\frac{2\pi}{L} rx\right) \right].$$

The Fourier series is discontinuous at $x_d = L/2 + nL$ for all integers n. By the theorem of discontinuity,

$$f(x_d) = \frac{1}{2} \left(\lim_{x \to x_d^+} f(x) + \lim_{x \to x_d^-} f(x) \right) = \frac{1}{2} \left(\frac{-L}{2} + \frac{L}{2} \right) = 0.$$

In-Class Exercises

In-Class Exercise 1.2.2. Find the Fourier series of the square wave shown below in Figure 1.4.

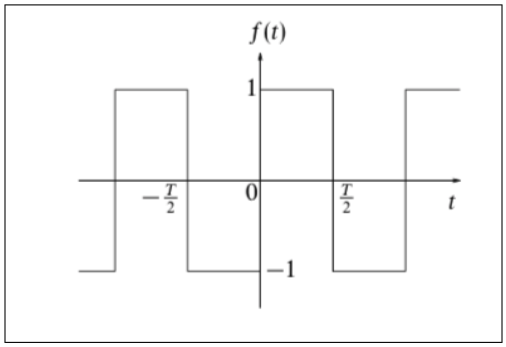


Figure 1.4: Square Wave [1]

In-Class Exercise 1.2.3. A triangular wave of period L is described by a periodic function $f(x) = |x|$ for $-L/2 \le x \le L/2$.

(a) Sketch the wave for $-2L \le x \le 2L$

(b) Is there any discontinuity?

(c) Let $L = 2\pi$, and find the Fourier series of $f(x)$.

(d) Discuss why the series does not contain sine components.

In-Class Exercise 1.2.4. Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \le \pi$. Hence, show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

(Exercise 12.5 from [1].)

1.3 Complex Fourier Series

We can use Euler's formula to write

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \text{ and } \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{1.13}$$

We can use these formulas to write the definitions of the hyperbolic sine and hyperbolic cosine:

$$\sinh(x) = -i\sin(ix) = \frac{e^x - e^{-x}}{2} \tag{1.14}$$

$$\cosh(x) = \cos(ix) = \frac{e^x + e^{-x}}{2} \tag{1.15}$$

Keeping this in mind, we can simplify the Fourier series to write the **complex Fourier series**:

$$f(x) = \sum_{r=-\infty}^{\infty} c_r e^{i\frac{2\pi}{L}rx}, \tag{1.16}$$

with

$$c_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-i\frac{2\pi}{L}rx}dx. \tag{1.17}$$

Example 1.3.1. Find the complex Fourier series for the wave defined by $f(x) = \cos\left(\frac{x}{2}\right)$ for $-\pi < x < \pi$.

Evaluate

$$c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(\frac{x}{2}\right)e^{-irx}dx = \frac{1}{2\pi}\frac{(-1)^r}{\frac{1}{4}-r^2}.$$

Then,

$$f(x) = \sum_{r=-\infty}^{\infty}\frac{1}{2\pi}\frac{(-1)^r}{\frac{1}{4}-r^2}e^{-irx}.$$

Reading Questions

Reading Question 1.3.2. Use Euler's formula to derive the expressions for the four trigonometric functions discussed above.

Reading Question 1.3.3. Use the Fourier series (Eq. 1.9) to derive for the complex Fourier series (Eq. 1.16).

1.4 The Fourier Transform

Suppose you receive a wave (or function) f that oscillates in time t. We now know that this wave can be expressed as an infinite-term linear combination of sinusoidal waves of different frequencies, and you are determined to find a distribution that displays the frequencies of the sinusoidal functions that summed to the wave you see now. In other words, we wish to take the wave from the domain of time, and transform it into the frequency domain so to see intensity versus frequency ω instead of intensity versus time. We also want to use this transformation on non-periodic functions. We will proceed with these motivation in mind.

Recall that

$$\int_{-\infty}^{\infty} g(\omega)\ d\omega = \lim_{\Delta k \to 0} \sum_{r=-\infty}^{\infty} g(r\Delta\omega)\Delta\omega, \tag{1.18}$$

(with $\omega = r\Delta\omega$). By substituting Eq. 1.17 into Eq. 1.16 (using t' as a variable of integration and T as the period of the function), we have that

$$f(t) = \sum_{r=-\infty}^{\infty} \left[\frac{1}{L} \int_{t_0}^{t_0+T} f(t') e^{-i\frac{2\pi}{T}rt'} dt' \right] e^{i\frac{2\pi}{L}rt}. \tag{1.19}$$

Let $t_0 = -T/2$ and define $\Delta\omega = 2\pi/T$ so,

$$f(t) = \sum_{r=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t') e^{-i\Delta\omega r(t'-t)}\ dt'. \tag{1.20}$$

Take $T \to \infty$ to account for non-periodic functions, so

$$f(t) = \lim_{T\to\infty} \sum_{r=-\infty}^{\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t') e^{-i\Delta\omega r(t'-t)}\ dt' \tag{1.21}$$

$$= \lim_{\Delta\omega\to 0} \sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} \int_{-\infty}^{\infty} f(t') e^{-i(r\Delta\omega)(t'-t)}\ dt' \tag{1.22}$$

$$= \lim_{\Delta\omega\to 0} \sum_{r=-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t') e^{-i\omega(t'-t)}\ dt' \right] \Delta\omega. \tag{1.23}$$

Using Eq. 1.18,

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t') e^{-i\omega(t'-t)} dt'\ d\omega \tag{1.24}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t') e^{-i\omega t'} dt' \right] e^{i\omega t}\ d\omega. \tag{1.25}$$

We define the **Fourier transform of** f as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\ dt, \tag{1.26}$$

and the **inverse Fourier transform of** $\tilde{f}$ as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t}\ d\omega. \tag{1.27}$$

It turns out that this does exactly what we set out to do! The Fourier transform takes a function in the time domain and brings it to the frequency domain. Figure 1.5 shows four sample functions along with the real part of their Fourier transform. This table should give an intuitive overview of how the Fourier transform works.

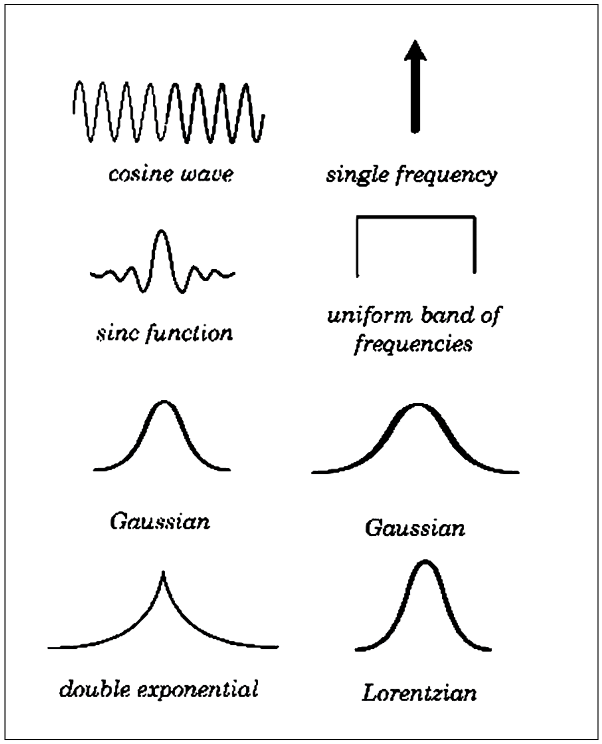
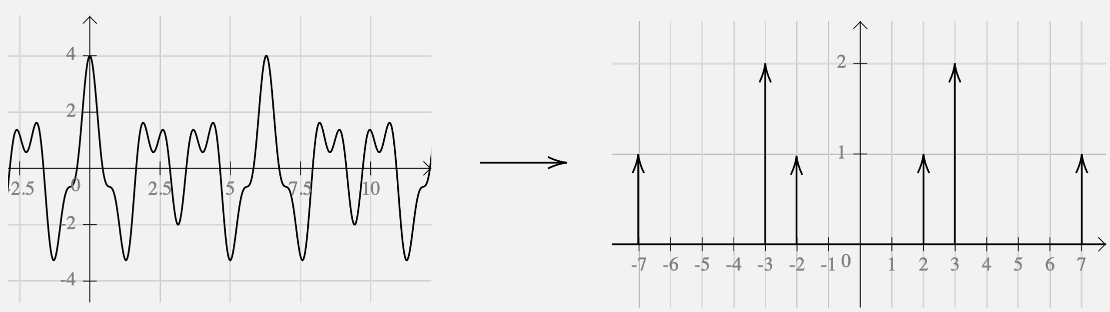


Figure 1.5: Sample Fourier Transforms

Example 1.4.1. A visual of the Fourier transform



The function in the time domain (on the left) is given by $f(t) = \cos(2t) + 2\cos(3t) + \cos(7t)$, so has component frequencies of 2, 3, and 7. We can see in how through the Fourier transform, the component frequencies are "separated" in the frequency domain on the right.

Consider a function f in the time (x) domain and let Δt denote the spread of f (e.g. the *Full Width at Half Max* or *FWHM* spread, which is the largest length W for which there is some l such that $f(l) = f(l + W) = \max(f)/2$. Similarly, let $\Delta\omega$ denote the spread of

$\tilde{f}(\omega)$. It is a general principle of the Fourier transform, that

$$\Delta t \Delta \omega \sim 1. \tag{1.28}$$

This happens to relate to the uncertainty principle in Quantum mechanics:

$$\Delta x \Delta p \sim \frac{h}{\pi}, \tag{1.29}$$

which is intuitive since deBroglie related a photon's frequency ν to its momentum p:

$$p = \frac{\nu}{c} h, \tag{1.30}$$

with the speed of light c and Planck's constant h.

Reading Questions

Reading Question 1.4.2. Explain Figure 1.5 qualitatively.

Reading Question 1.4.3. Find the Fourier transform of $f(x) = e^{-a|x|}$ for $a > 0$. Then sketch $f(x)$ and $\tilde{f}(k)$. Discuss the two graphs qualitatively using Equation 1.28.

1.5 The Dirac Delta Function

The **Dirac Delta function** (or just **Delta function**) is sometimes referred to as the 'unit impulse symbol' and is very useful in modeling point charge, point mass, etc. The δ function is an infinite spike that encloses an area of 1 and is defined as

$$\delta(x - c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-c)}\, dk. \tag{1.31}$$

The value of the function at $x \neq c$ is 0, and is 'infinity' at $x = c$ so graphically the delta function is represented by a horizontal line and an arrow at $x = c$. Generally, the height of the arrow is representative of any multiplicative constant. See Figure 1.6 for an example of this.

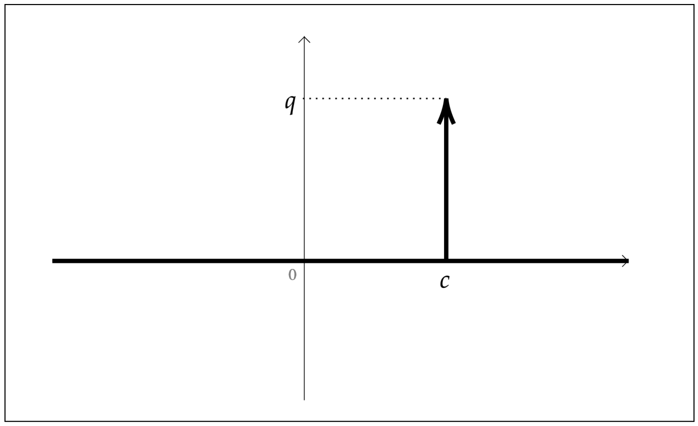


Figure 1.6: The Dirac Delta, $q\delta(x-c)$

The following are some useful properties of the Delta function.

$$\delta(-x) = \delta(x) \tag{1.32}$$
$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{1.33}$$
$$x\delta(x) = 0 \tag{1.34}$$

The Dirac Delta can be used to 'select' the value of the function at $x = c$ since the value of $\delta(x)$ is zero everywhere except $x = c$. This is done with integration:

$$\int_a^b f(x)\delta(x-c)\ dx = \begin{cases} f(c) & \text{if } a < c < b; \\ 0 & \text{otherwise.} \end{cases} \tag{1.35}$$

The same is true when a approaches infinity or b approaches negative infinity. This may be intuitive with a picture (see reading questions), but the case where $a \to \infty$ and $b \to -\infty$ it can be shown directly using the Fourier transform:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\delta(x-c)\ dx &= \int_{-\infty}^{\infty} f(x)\delta(c-x)\ dx && (1.36)\\
&= \int_{-\infty}^{\infty} f(x)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(c-x)}\ dk\ dx && (1.37)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} e^{-ikx}e^{ikc}\ dk\ dx && (1.38)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx\ e^{ikc}\ dk && (1.39)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikc}\ dk && (1.40)\\
&= f(c). && (1.41)
\end{aligned}$$

It is interesting also to note that the Fourier transform of $\delta(t)$ yields

$$\widetilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt = \frac{e^0}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}, \tag{1.42}$$

so in the frequency domain, $\widetilde{\delta}(t)$ is a horizontal line at height $\frac{1}{\sqrt{2\pi}}$. This is what we expect since by Eq. 1.28, $\Delta t \to 0$ in the time domain implies that $\Delta\omega \to \infty$ in the frequency domain.

Reading Questions

Reading Question 1.5.1. Prove that $\delta(-x) = \delta(x)$.

Reading Question 1.5.2. Draw a picture illustrating Eq. 1.35.

In-Class Exercises

In-Class Exercise 1.5.3. Prove the following properties of Dirac delta:

- $\delta(ax) = \frac{1}{|a|}\delta(x)$
- $x\delta(x) = 0$

In-Class Exercise 1.5.4. Evaluate the following integrals:

(a) $\int_2^6 (3x^2 - 2x - 1)\, \delta(x-3)\ dx$

(b) $\int_0^5 \delta(x - \pi) \cos x\ dx$

(c) $\int_0^3 x^3 \delta(x+1)\ dx$

(d) $\int_{-\infty}^{\infty} \delta(x+2) \ln(x+3)\ dx$

(e) $\int_{-\infty}^{1} 9x^2 \delta(3x+1)\ dx$

(f) $\int_{-1}^{1} 9x^2 \delta(3x+1)\ dx$

(g) $\int_{-\infty}^{\infty} \delta(x-b)\ dx$

1.6 The Laplace Transform

A **complex function** is a function whose output can be complex. While the Fourier transform of a function is a complex function of a real variable (frequency, ω), the **Laplace transform** of a function is a complex function of a complex variable (complex frequency, s). We denote the Laplace transform of a function f by either $\mathcal{L}[f(t)]$ or $\bar{f}(s)$, and in fact, the Laplace transform can be defined with the Fourier transform:

$$\bar{f}(s) = \mathcal{L}\left[f(t)\right] = \int_0^{\infty} f(t) e^{-st}\ dt = \sqrt{2\pi}\widetilde{f}(-is). \tag{1.43}$$

It turns out that while similar to the Fourier transform, the Laplace transform has some different but interesting uses but is restricted to only $t \geq 0$, so we will only consider 'causal' functions:

$$f(x) = \begin{cases} \text{nonzero in general} & t \geq 0 \\ 0 & t < 0. \end{cases} \tag{1.44}$$

Example 1.6.1. Find the Laplace transform of f given by $f(t) = e^{at}$ for $t \geq 0$.

$$\mathcal{L}[f(t)] = \int_0^\infty e^{at} s^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt = \lim_{i\to\infty} \left[\frac{-e^{-(s-a)t}}{s-a} \right]_{t=0}^{t=i} = \frac{1}{s-a}, \tag{1.45}$$

for $s > a$.

The Laplace transform is linear, since given any function f given by $f(t) = c_1 f_1(t) + c_2 f_2(t)$, with scalars c_1 and c_2, and functions f_1 and f_2,

$$\mathcal{L}\left[f(t)\right] = c_1 \mathcal{L}\left[f_1(t)\right] + c_2 \mathcal{L}\left[f_2(t)\right]. \tag{1.46}$$

Example 1.6.2. Find the Laplace transform of f given by $f(t) = t^n$ for $t \geq 0$ and $n = 1, 2, 3, \ldots$.

Recall that by a Taylor series expansion,

$$\mathcal{L}[e^{at}] = \mathcal{L}\left[\sum_{n=0}^\infty \frac{a^n}{n!} t^n\right] = \sum_{n=0}^\infty \mathcal{L}\left[\frac{a^n}{n!} t^n\right] = \sum_{n=0}^\infty \frac{a^n}{n!} \mathcal{L}\left[t^n\right], \tag{1.47}$$

and by our previous example and the geometric series ($\sum_{n=0}^\infty x^n = \frac{1}{1-x}$ for $x < 1$),

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} = \frac{1/s}{1-\frac{a}{s}} = \frac{1}{s} \sum_{n=0}^\infty \left(\frac{a}{s}\right)^n = \sum_{n=0}^\infty \frac{a^n}{s^{n+1}}. \tag{1.48}$$

This implies that

$$\sum_{n=0}^\infty \frac{a^n}{n!} \mathcal{L}\left[t^n\right] = \sum_{n=0}^\infty \frac{a^n}{s^{n+1}}, \tag{1.49}$$

so

$$\sum_{n=0}^\infty a^n \left(\frac{\mathcal{L}\left[t^n\right]}{n!} - \frac{1}{s^{n+1}} \right) = 0. \tag{1.50}$$

Thus, for all $a \neq 0$,

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}, \tag{1.51}$$

for $n = 1, 2, 3, \ldots$.

The **inverse Laplace transform** of a function $\bar{f}$ is written $\mathcal{L}^{-1}[\bar{f}(s)]$ and also happens to be linear, so for any function $\bar{f}$ given by $\bar{f}(t) = c_1\bar{f}_1(t) + c_2\bar{f}_2(t)$, with scalars c_1 and c_2, and functions $\bar{f}_1$ and $\bar{f}_2$,

$$\mathcal{L}^{-1}\left[\bar{f}(t)\right] = c_1\mathcal{L}^{-1}\left[\bar{f}_1(t)\right] + c_2\mathcal{L}^{-1}\left[\bar{f}_2(t)\right]. \tag{1.52}$$

We will not compute the inverse Laplace transform directly, so for ease, Table 1.1 summarizes many useful Laplace transform relations. The transformations are valid for $s > s_0$.

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/S^{n+1}$	0
$\sin bt$	$b/\left(s^2+b^2\right)$	0
$\cos bt$	$S/\left(s^2+b^2\right)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/\left(s^2-a^2\right)$	$\|a\|$
$\cosh at$	$s/\left(s^2-a^2\right)$	$\|a\|$
$e^{at}\sin bt$	$b/\left[(s-a)^2+b^2\right]$	a
$e^{at}\cos bt$	$(s-a)/\left[(s-a)^2+b^2\right]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta\left(t-t_0\right)$	e^{-st_0}	0
$H\left(t-t_0\right) = \begin{cases} 1 & \text{for } t \geq t_0 \\ 0 & \text{for } t < t_0 \end{cases}$	e^{-st_0}/s	0

Table 1.1: Standard Laplace transforms, valid for $s > s_0$ [1]

It is also useful to consider the Laplace transform of the derivative of a function f. In other words, we want to find $\mathcal{L}[f'(t)]$. We can use the product rule as follows:

$$\begin{aligned}
\mathcal{L}[f'(t)] &= \int_0^\infty f'(t)e^{-st}\ dt &(1.53)\\
&= \int_0^\infty (f(t)e^{-st})' - f(t)(e^{-st})'\ dt &(1.54)\\
&= \int_0^\infty (f(t)e^{-st})' - \int_0^\infty f(t)(e^{-st})'\ dt &(1.55)\\
&= [f(t)e^{-st}]_0^\infty + \int_0^\infty sf(t)e^{-st}\ dt &(1.56)\\
&= \lim_{t\to\infty}[f(t)e^{-s(t)}] - f(0)e^0 + s\int_0^\infty f(t)e^{-st}\ dt &(1.57)\\
&= 0 - f(0) + s\bar{f}(s), &(1.58)
\end{aligned}$$

for $s > 0$. The Laplace transforms of some higher derivatives are

$$\mathcal{L}[f'(t)] = s\bar{f}(s) - f(0), \tag{1.59}$$
$$\mathcal{L}[f''(t)] = s^2\bar{f}(s) - (sf(0) + f'(0)), \tag{1.60}$$
$$\mathcal{L}[f'''(t)] = s^3\bar{f}(s) - (s^2 f(0) + sf'(0) + f''(0)), \tag{1.61}$$
$$\mathcal{L}[f''''(t)] = s^4\bar{f}(s) - (s^3 f(0) + s^2 f'(0) + sf''(0) + f'''(0)). \tag{1.62}$$

The pattern is fairly obvious and it can be proved inductively that in general,

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n\bar{f}(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0). \tag{1.63}$$

The Laplace transform and inverse transform can be used with the forms for Laplace transforms of derivatives quite simply to solve rather complicated differential equations. First, however, it is important to discuss a routine procedure that becomes quite useful.

Partial fraction decomposition is the way in which a rational expression with a polynomial denominator can be written as a sum of rational expressions with denominators of a lower degree. The general procedure can be summarized in 4 steps:

1. Factor the denominator of the original rational expression as far as possible.
2. Set the original expression equal to the sum of appropriate terms (see Table 1.2).
3. Write the sum as a single rational expression by finding a common denominator.
4. Use the two numerators to solve a system of equations for the unknown values, A, B, C, etc.

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax+b}$
$(ax + b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Table 1.2: Some Terms for Partial Fraction Decomposition

While Table 1.2 is not exhaustive, it covers the range of complexity that will be required in this book.

To solve differential equations using the Laplace and inverse Laplace transforms, first take the Laplace transform of the differential equation. Then, use the necessary pre-existing conditions to solve for the function.

Example 1.6.3. Solve the differential equation $f'' - 2f' + f = t$ with initial conditions $f(0) = 0$ and $f'(0) = 0$, using the Laplace and the inverse Laplace transform.

First, take the Laplace transform of the equation and solve for $\bar{f}(s)$:

$$\begin{aligned} &\mathcal{L}[f'' - 2f' + f] = \mathcal{L}[t] && (1.64)\\ \implies &\mathcal{L}[f''] - \mathcal{L}[2f'] + \mathcal{L}[f] = \mathcal{L}[t] && (1.65)\\ \implies &s^2\bar{f}(s) - (sf(0) + f'(0)) - 2(s\bar{f}(s) - f(0)) + \bar{f}(s) = \frac{1}{s^2} && (1.66)\\ \implies &\bar{f}(s)(s^2 - 2s - 1) = \frac{1}{s^2} && (1.67)\\ \implies &\bar{f}(s) = \frac{1}{s^2(s-1)^2}. && (1.68) \end{aligned}$$

In order to make taking the inverse Laplace transform simple, we can rewrite $\bar{f}(s)$ using partial fraction decomposition. Since the denominator is already factored, write

$$\begin{aligned} \bar{f}(s) = \frac{1}{s^2(s-1)^2} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} && (1.69)\\ &= \frac{As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2}{s^2(s-1)^2} && (1.70)\\ &= \frac{(A+C)s^3 + (-2A + B - C + D)s^2 + (A - 2B)s + B}{s^2(s-1)^2}. && (1.71) \end{aligned}$$

This implies that we must solve the following system of equations:

$$\begin{cases} 0 = A + C \\ 0 = -2A + B - C + D \\ 0 = A - 2B \\ 1 = B \end{cases} \tag{1.72}$$

We find that $A = 2$, $B = 1$, $C = -2$, and $D = 1$ so we can use Table 1.2 can write

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\left[\bar{f}(s)\right] &= \mathcal{L}^{-1}\left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] && (1.73)\\ &= \mathcal{L}^{-1}\left[\frac{2}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] - \mathcal{L}^{-1}\left[\frac{2}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] && (1.74)\\ &= t + 2 - 2e^t + te^t && (1.75)\\ &= t + 2 + (t-2)e^t. && (1.76) \end{aligned}$$

Reading Questions

Reading Question 1.6.4. Use Eq. 1.43 to prove that $\mathcal{L}$ is linear (Eq. 1.46).

In-Class Questions

In-Class Exercise 1.6.5. Verify the Laplace transforms given in Table 1.1 for $\sin bt$, $\cos bt$, $\sinh at$, $\cosh at$, $e^{at}\sin bt$, and $e^{at}\cos bt$.

In-Class Exercise 1.6.6. Find $f(t)$ for the following Laplace transforms:

(a) $\overline{f}(s) = \frac{4}{s^2-3s}$

(b) $\overline{f}(s) = \frac{s+3}{s(s+1)}$

(c) $\overline{f}(s) = \frac{2s+1}{s^2+3s-10}$

(d) $\overline{f}(s) = \frac{2s+3}{s^2+4}$

In-Class Exercise 1.6.7. Use Eq. 1.59 to prove the formula for $\mathcal{L}[f''(t)]$.

In-Class Exercise 1.6.8. Solve $f''(t) + 2f'(t) - 3f(t) = e^t$ for $f(0) = 0$ and $f'(0) = 1$.

1.7 Exercises

Exercise 1.7.1. Find the Fourier series and their values at discontinuities, if any, for the following periodic functions:

(a) $f(x) = \begin{cases} 0 & \text{for } -1 \le x < 0 \\ 1-x & \text{for } 0 \le x < 1. \end{cases}$

(b) $f(x) = |\sin(x)|$ for $-\pi/2 < x < \pi/2$.

(c) $f(x) = \begin{cases} x^2 & \text{for } -1 \le x < 0 \\ 0 & \text{for } 0 \le x < 1. \end{cases}$

Exercise 1.7.2. Find complex Fourier series for the following functions:

(a) $f(x) = x$ for $-\pi \le x \le \pi$

(b) $f(x) = |x|$ for $-\pi \le x \le \pi$

(c) $f(x) = \cos\frac{x}{2}$ for $-\pi \le x < \pi$

Exercise 1.7.3. Edit `FourierSeries.py` (Listing 5.1) and then print out the graph. (See the **Listings** for instructions on installing and using Python).

Exercise 1.7.4. Express the following in partial fraction form (Exercise 1.16 from [1]):

(a) $\frac{2x^3-5x+1}{x^2-2x-8}$,

(b) $\frac{x^2+x-1}{x^2+x-2}$.

Exercise 1.7.5. Resolve the following into partial fractions in such a way that x does not appear in any numerator (Exercise 1.17 from [1]):

(a) $\frac{x^3+3x^2+x+19}{x^4+10x^2+9}$,

(b) $\frac{x-6}{x^3-x^2+4x-4}$.

Exercise 1.7.6. Resolve the following into partial fractions in such a way that x does not appear in any numerator (Exercise 1.18 from [1]):

(a) $\frac{2x^2+x+1}{(x-1)^2(x+3)}$,

(b) $\frac{x^2-2}{x^3+8x^2+16x}$,

(c) $\frac{x^3-x-1}{(x+3)^3(x+1)}$.

Exercise 1.7.7. Find the functions $y(t)$ whose Laplace transforms are the following (Exercise 13.22 from [1]):

(a) $1/\left(s^2-s-2\right)$;

(b) $2s/\left[(s+1)\left(s^2+4\right)\right]$;

(c) $e^{-(y+s)t_0}/\left[(s+\gamma)^2+b^2\right]$.

Exercise 1.7.8. Use the method of Laplace transforms to solve (Exercise 15.10 from [1]):

(a) $\frac{d^2f}{dt^2}+5\frac{df}{dt}+6f=0, \quad f(0)=1, f'(0)=-4,$

(b) $\frac{d^2f}{dt^2}+2\frac{df}{dt}+5f=0, \quad f(0)=1, f'(0)=0.$

Chapter 2

Vector Calculus

2.1 Vector Algebra

2.2 Differentiation and Integration of Vectors

2.3 Operators

2.4 Non-Cartesian Coordinates

2.5 Integration of Vector Fields

2.6 Integral Theorems

2.7 Exercises

Chapter 3

Linear Algebra

Chapter 4

Partial Differential Equations

Chapter 5

Listings

This chapter holds the various Python scripts referred to throughout the book. A good resource to become familiar with coding in Python is

`https://www.w3schools.com/python/python_intro.asp`.

To install Anaconda Python distribution on your computer, follow the instruction at `https://docs.anaconda.com/anaconda/install/` and make sure to install the Python 3.7 version. When editing and creating scripts, use the `Jupyter` Notebook included in the distribution.

5.1 FourierSeries.py

```python
# -*- coding: utf-8 -*-
from numpy import *
from matplotlib.pyplot import *

#----- Set period and x0 ------
L =
x0 =

#----- Set maximum value for r (r = 1, 2, ..., N)
N =

#---- Define function and its fourier components --------
def f(x):
    return

a0 =

def a(r):
    return

def b(r):
    return

#-------------------------------------------------------
# Change up to here only!
#-------------------------------------------------------
def FourierSeries(x, N):
    value = a0 / 2
    for r in range(1, N + 1):
        value += a(r) * cos(2 * pi * r * x / L)
        value += b(r) * sin(2 * pi * r * x / L)
    return value
x = linspace(x0, x0 + L, 1000)
y1 = f(x)
y2 = FourierSeries(x, N)

#------ Graphics -------------------------------------
figure(figsize=(10, 8))
xlim(x0, x0+L)
plot(x, y1, 'b', label=r'$f(x)$')
plot(x, y2, 'r', label='Fourier Sereis')
text(x0+L/2, y1.max()*4/5, 'N='+str(N), fontsize=20)
grid(True)
legend(loc='best', fontsize=12)
show()
```

Selected Hints and Solutions

Fourier Analysis

1.1.2: Hint: What can you say if 4 divides n? What about if $n/4$ leaves a remainder of 1, 2, or 3?
1.1.9: (a) $5+3i$; (b) $-1-5i$; (c) $10+5i$; (d) $2/5+11i/5$; (e) 4; (f) $3-4i$; (g) $\ln 5 + i\,[\tan^{-1}(4/3)+2n\pi]$; (h) $\pm(2.521+0.595i)$
1.1.10: (a) $2e^{i5\pi/6}$; (b) $2\sqrt{2}e^{-i\pi/4}$; (c) $(3/2)e^{-i\pi/2}$; (d) $e^{-i2\pi/3}$; (e) $e^{i\pi/2}$; (f) $2e^{i\pi}$
1.2.2: $f(t)=\frac{4}{\pi}\left(\sin\omega t+\frac{\sin 3\omega t}{3}+\frac{\sin 5\omega t}{5}+\cdots\right)$, with $\omega = 2\pi/T$, the angular frequency.
1.4.3: Hint: what is the behavior of each as a changes?
1.5.4 (a) 20; (b) -1; (c) 0; (d) 0; (e) 1; (f) 1/3; (g) 1
1.6.6: (a) $-4/3\,(1-e^{3t})$; (b) $3-2e^{-t}$; (c) $(5/7)e^{2t}+(9/7)e^{-5t}$; (d) $(3/2)\sin 2t + 2\cos 2t$
1.6.8: Hint: Use $\frac{s}{(s+a)(s+b)^2}=\frac{A}{s+a}+\frac{B}{s+b}+\frac{C}{(s+b)^2}$. Answer: $(1/4)te^t+(3/16)e^t-(3/16)e^{-3t}$

Bibliography

[1] K. F. Riley, M. P. Hobson, S. J. Bence, *Mathematical Methods for Physics and Engineering*, Cambridge, third edition.

[2] Benjamin Kennedy, *Introduction to Real Analysis*, Gettysburg College, 2018 edition.