How explicit is the Explicit Formula?

Barry Mazur and William Stein

June 5, 2018

1 Introduction

The term 'Explicit Formula' as in our title refers to the genre of formula that expresses *arithmetically interesting quantities* in terms of the zeroes of a related zeta-function or L-function. The first such formula appears towards the end of Bernhard Riemann's short and great paper "Über die Anzahl der Primzahlen unter einer gegebenen Grösse," published in the Monatsberichte der Berliner Akademie, November 1859. For real numbers $X > 1$ not equal to a power of a prime number (and with some minor changes in notation and ordering of terms) Riemann's formula is:

$$(*) \quad \sum_n \frac{1}{n}\pi(X^{\frac{1}{n}}) \quad = \quad \mathrm{Li}(X) \quad + \quad \left\{\int_X^\infty \frac{1}{x^2-1}\cdot\frac{dx}{x\log(x)} + \log(\xi(0))\right\} \quad - \quad \sum_\theta \left(\mathrm{Li}(X^{\frac{1}{2}+i\theta}) + \mathrm{Li}(X^{\frac{1}{2}-i\theta})\right).$$

Here, the Riemann Hypothesis is assumed, the $\pm\theta$ are the imaginary parts of the nontrivial zeroes, and $\xi(0)$ is—it seems—a scribe's error for $\zeta(0) = 1/2$. The three terms on the right correspond to, respectively: the pole of $\zeta(s)$ at $s = 1$, the trivial zeroes of $\zeta(s)$, and the nontrivial zeroes of $\zeta(s)$.

[[William: we need to address the fact that the above formula is somehow nonsense, right? See worksheets/2016-06-08-150036-very-explicit-zeta-formula.sagews]]

Subsequently, there have been many versions of Explicit Formulas, one such notable variant proved by Hans von Mangoldt, in his 1895 article "Zu Riemann's Abhandlung 'Über die Anzahl der Primzahlen unter einer gegebenen Grösse'", Journal für die reine und angewandte Mathematik.

In celebration of Riemann's formula we will discuss some computations and open questions related to the application of the Explicit Formula to the arithmetic of elliptic curves. This follows constructions in a letter of Peter Sarnak to one of the authors ([**?**]).

A cartoon version of the genre of 'Explicit Formula' might be described as follows:

(**) *Sum of local data* $=$ *Global datum* $+$ *Easy error term* $+$ *Oscillatory term*,

where each term of the formula is taken as a function of a 'cutoff' real value X, and the ordering of the terms is in accord with **(*)** above. The *Sum of local data* that appears in most of the 'Explicit Formulas' of analytic number theory is often given, i.e., by a partial sum $\nu(X) \cdot \sum_{p<X} G(p)$ of locally defined *arithmetically interesting* quantities $G(p)$ attached to prime numbers p, summed to up to some cutoff value $p < X$ (and normalized for convenience by an elementary (continuous) multiplicative factor $\nu(X)$).

Usually the *Global datum* is computed by knowing the order of specific zeroes (or poles) at specified values of the complex variable s of relevant (global) L-functions. For example, the formula (*) gives the contribution of the pole at $s = 1$ of $\zeta(s)$. Often (perhaps only conjecturally) the *Global datum* is the 'dominant term' on the right-hand-side of the equation. Sometimes, as will be the case of the examples to be described below, the roles are reversed and one takes the Global datum as the object being studied. That is, we view it as *Global datum* $=$ *Sum of local data* $-$ *Easy error term* $-$ *Oscillatory term*.

In some instances, the sought-for Global datum is (conjecturally) constant, independent of the cutoff X—in fact, the mean of *Sum of local data*—and also an integer, in which case a close approximation to each of the other three terms (for some specific value of X) would give a decisive answer for the value of the Global datum.

1.1 An example:

Consider a non-CM elliptic curve E over $\mathbf{Q}$ of conductor N uniformized by a newform f_E (of conductor N and of weight two) with Fourier expansion $f_E(q) = \sum_{n\geq 1} a_n(E)q^n$. Form the following *Sum of local data*:

$$\mathcal{D}_E(X) := \frac{\log\ X}{\sqrt{X}} \sum_{p\leq X} \frac{a_E(p)}{\sqrt{p}},$$

where (assuming the Riemann Hypothesis for the L-function attached to f_E) the explicit formula gives us:

$$\mathcal{D}_E(X) \ = \ 1 - 2r_E \ + \ \textit{Easy error term} \ + \sum_{|\gamma|\leq T} \frac{X^{i\gamma}}{\frac{1}{2} + i\gamma}$$

[[TODO: explain T, e.g., take limit as $T \to \infty$ or make error term depend on T.]]

Here, the 'global datum' is $1 - 2r_E$ where r_E is the analytic rank of E (i.e., the order of vanishing of

$L(E,s)$ at the central point; conjecturally this will be the Mordell-Weil rank of E). The summation is taken for γ ranging over the imaginary parts of the nontrivial zeroes of the L-function attached to E.

The values of $\mathcal{D}_E(X)$ achieve a limiting distribution μ_E (with respect to multiplicative Haar measure dx/x) with *mean* M equal to $1-2r_E$ and variance V equal to

$$\lim_{T\to\infty} \sum_{|\gamma|\leq T} \frac{1}{\frac{1}{4}+\gamma^2}.$$

Under further conjectures ([**?**]) the measure μ_E is symmetric about its mean $1-2r_E$, it is smooth, and has support all of $(-\infty,+\infty)$. It is natural to view this in the context of Chebyshev biases; i.e., to interpret—when r_E is positive—the minus sign in $-r_E$ as a measure of the bias that the values $a_p(E)$ have to being negative. This type of *bias* in the arithmetic statistics of elliptic curves hearkens back to the early work of Birch and Swinnerton-Dyer [**?**], and in the context we are considering was first written down by Peter Sarnak in the spirit of the classical Chebyschev bias, and 'prime races.' An 'Explicit Formula' account of this in the context of prime numbers in different arithmetic progressions can be found in [**?**]. See also [**?**], [**?**] which fits into this Explicit Formula format, as does work of Martin-Watkins, Fiorilli, Bober-Mestre-Odlyzko, Conrey-Snaith, Spicer, and others.

For this volume in honor of Bernhard Riemann, we will offer computations related to this, and one other interesting case that does depend on these 'finer conjectures', advertise the conjectures themselves, and the need for computational projects regarding such problems. We restrict ourselves to applications to elliptic curves, but we take this only as a basic (important) example of a fuller story for general motives. We have future plans for a web-accessible resource[1]: a repository of some of the numerics for the cases related to elliptic curves that interest us.

One problem we consider is related to the question—given an elliptic curve over the rational numbers and letting p range through prime numbers—of how often $p+1$ is an *over-count* or an *under-count* for the number of rational points on the curve modulo p? The rough answer is 50/50, but there can be a 'bias'.

We offer no new theoretical results but, as mentioned, we use this occasion to exhibit computations and recall some interesting recent work and conjectures (of other people) that might warrant more such computations and that raise a host of questions, both theoretical and computational. For example, to do some systematic numerical computations related to an elliptic curve E attached to a newform f_E (along the lines of what has already been done in this paper) it would be very useful to have a much larger data set of the arithmetic function

$$n \mapsto r_E(n)$$

where $r_E(n)$ is the order of vanishing at $s=1$ of the L-function of the automorphic forms $\mathrm{symm}^n f_E$ for odd values of n. Regarding this arithmetic function, aside from having control of the parity of $r_E(n)$ (e.g., see [**?**]) hardly anything else is known. Nor do we (at least, the authors of this paper) yet have enough experience—when E has no complex multiplication—even to formulate a proper conjecture.

[1] See `http://wstein.org/papers/2016-explicit/`

We might also mention that when making these numerical experiments one seems to be in a situation that is not entirely dissimilar from the type of slightly annoying mismatch between conjecture and data that one encounters in more traditional studies of Mordell-Weil statistics that was the subject of the survey article [**?**], and the more recent [**?**]. But this may be unavoidable, given that even the so-called 'easy error term' in the explicit formula may tend to zero rather slowly.

We should say at the outset that for simplicity, and sometimes for necessity, we'll be assuming GRH throughout—without any further mention. In fact, at times we'll also be assuming (*with* explicit warning) some further conjectures. Given our current state of knowledge, it would be interesting enough to work conditionally on some reasonable conjectures that include or extend the Grand Riemann Hypothesis. In fact, the search for an answer to such questions might give motivation for the refinement of conjectures—interesting in their own right—that complement the Riemann Hypothesis.

2 Sarnak distributions for the elliptic curve E relative to the weighting function V

In a letter [**?**] to one of us (to B.M.) Peter Sarnak considered a broad array of statistics related to local arithmetic data associated to modular forms, and, in particular, to elliptic curves over $\mathbf{Q}$.

For p a prime, write

$$\frac{a_E(p)}{\sqrt{p}} := \alpha_p + \beta_p, \tag{2.1}$$

with $\alpha_p = e^{i\theta_p}$ and $\beta_p = e^{-i\theta_p}$ and

$$\theta_p \in [0, \pi]. \tag{2.2}$$

Our basic data consists of the function

$$p \mapsto \theta_p \tag{2.3}$$

To have some vocabulary to deal with its statistics, consider

$$U_n(\theta) := \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

so we have:

$$\frac{a_E(p)}{\sqrt{p}} = U_1(\theta_{E,p}).$$

Note that the set $\{U_n\}$ for $n = 0, 1, 2, \ldots$ forms an orthonormal basis of the Hilbert space $L^2[0, \pi]$ with the inner product

$$\langle f, g\rangle := \frac{2}{\pi}\int_0^\pi f(\theta)g(\theta)\sin^2(\theta)d\theta.$$

For $V(\theta)$ a smooth function on $[0,\pi]$, write $V = \sum_{n=0}^\infty c_n U_n$ with $c_n := \langle V, U_n\rangle$.

Let us define the "V-weighted average of the data"

$$D_V(X) := \frac{\log X}{\sqrt{X}} \sum_{p\le X} V(\theta_p) \tag{2.4}$$

so that

$$\mathcal{D}_E(X) = \mathcal{D}_{U_1}(X).$$

Just to cut down to the essence as rapidly as possible, and just for the present paper:

Definition 2.1. Say that our data (2.3) has **'Explicit Formula' statistics** if there is a sequence of non-negative integers $\{r_n\}_n$ for $n = 1, 2, 3, \dots$ such that for all smooth functions $V(\theta)$ as above with $c_0 = 0$,

- possesses a limiting distribution[2] μ_V with respect to the multiplicative measure dX/X,
- μ_V has support on all of $\mathbf{R}$, is continuous and symmetric about its mean, $\mathcal{E}(S_V)$, and

$$\mathcal{E}(S_V) \;=\; -\sum_{n=1}^\infty c_n\big(2r_n + (-1)^n\big). \tag{2.6}$$

We will refer to μ_V as the **Sarnak Distribution (for the elliptic curve E relative to the weighting function V)**. One can also compute—given some plausible conjectures—the behavior of the **variance** (i.e., the measure of fluctuation of the values of $S_V(X)$ about the mean) as well; the variance is defined by the formula

$$\mathcal{V}(S_V) := \mathcal{E}\big([S_V - \mathcal{E}(S_V)]^2\big).$$

Remark 2.2. If some standard conjectures[3] and some non-standard conjectures[4] hold, then our

[2] Recall that, as in subsection **??** above, $S_V(x)$ **possesses a limiting distribution** μ_V **with respect to the multiplicative measure** dx/x if for continuous bounded functions f on $\mathbf{R}$ we have:

$$\lim_{X\to\infty} \frac{1}{\log X}\int_0^X f(S_V(x))dx/x \;=\; \int_{\mathbf{R}} f(x)d\mu_V(x). \tag{2.5}$$

[3]that (for $n = 1, 2, \dots$) the L-functions of the symmetric n-th powers of the elliptic curve,

$$L(s, E, \mathrm{sym}^n) := \prod_p \prod_{j=0}^n (1 - \alpha_p^{n-j}\beta_p,^j\, p^{-s})^{-1}, \tag{2.7}$$

have analytic continuation to the entire complex plane satisfying a standard function equation (and one can relax analyticity and require merely an appropriate meromorphicity hypothesis) and that they be holomorphic and non-vanishing up to $Re(s) = 1/2$ (i.e., GRH). The integer r_n (for $n = 1, 2, \dots$) is then the multiplicity of the zero of $L(s, E, \mathrm{sym}^n)$ as $s = 1/2$.

[4]LI(E); see **??**, **??** [[TODO:expand on this mysterious note – addressed later "Fiorilli calls his..."]]

data (2.3) would indeed have *'Explicit Formula' statistics*; for details, see [?]. The integers r_n, which by footnote are (conjecturally) the orders of vanishing of specific L-functions at their central points, are expected to have the large preponderance of their values equal to 0 or 1, depending on the sign of the functional equation satisfied by the L-function to which they are associated, so the *mean* for a given V as computed by equation (2.6) stands a good chance of being finite.

3 The Letter of Peter Sarnak

In a letter [?] to one of us (to B.M.) Peter Sarnak sketched reasons for the statements made about the two formats for sums of local data that we introduced above, and indeed, for large class of such formats. As we understand it, the computations in that letter was, at least in part, the fruit of conversations with Andrew Granville and also an outgrowth of [?]. We are grateful for that letter, and for illuminating discussions with Granville, Rubinstein, and Sarnak. Assuming a list of standard conjectures about the behavior of L-functions, together with some very plausible but less standard conjectures, Sarnak begins by showing—as we mentioned above—that (conditional on standard conjectures) the following 'Sum of local data'

$$\mathcal{D}_E(X) := \frac{\log\ X}{\sqrt{X}} \sum_{p\leq X} \frac{a_{\mathcal{E}}(p)}{\sqrt{p}},$$

has a limiting distribution with *mean* equal to $1 - 2r_E$; the *variance* of this limiting distribution is the sum of the squares of the reciprocals of the absolute values of the nonreal zeroes of the L-function of E. The argument for these (and related) facts follows Mike Rubenstein's and Peter Sarnak's line of reasoning in the article *Chebyshev's Bias* [??]. For another expository account of number theoretic issues related to biases, see [??]. Similar reasoning works for other formats, including the *raw* sum of local data as will be depicted in our graphs below; i.e.,

$$\Delta_E(X) := \frac{\log\ X}{\sqrt{X}} \big(\#\{p \leq X;\ a_E(p) > 0\}\ -\ \#\{p \leq X;\ a_E(p) < 0\}\big),$$

which (given reasonable conjectures, and guesses) one discovers to have infinite *variance* so whatever bias we will be seeing in our finite stretch of data will eventually wash out[5].

[[never use (*) for formulas; let latex do its thing)]]

The bias of undercounts versus overcounts ([?]):

Regarding 'biases,' thanks to the recent resolution [] [[todo: in https://webusers.imj-prg.fr/ michael.harris/SatoTate of the Sato-Tate Conjecture in this context, one knows that—roughly—half the Fourier coefficients $a_E(p)$ are positive and half negative (for non-CM curves). That is, the ratio

$$\frac{\#\{p < X \mid N_E(p) < p+1\}}{\#\{p < X \mid N_E(p) > p+1\}} \ = \ \frac{\#\{p < X \mid a_E(p) > 0\}}{\#\{p < X \mid a_E(p) < 0\}}$$

[5] All this is specific to elliptic curves E with no complex multiplication, as our examples below all are. The non-finiteness of the variance is related to the fact that the (expected) number of zeroes—in intervals $(1/2, i/2 + iT)$ $(T > 0)$—of the L function of the n-th symmetric power of the newform f_E attached to E grows at least linearly with n.

tends to 1 as X goes to infinity. Moreover, the numbers of positive values and negative values look very close to each other:

Curve	Rank	Negative $a_E(p)$ for $p < 10^9$	Positive $a_E(p)$ for $p < 10^9$	Difference
11a	0	25422268	25423101	-833
14a	0	25422229	25421074	1155
128b	0	25420641	25425608	-4967
816b	0	25424848	25421229	3619
2379b	0	25417900	25427007	-9107
5423a	0	25420479	25425242	-4763
29862s	0	25420525	25425197	-4672
37a	1	25423396	25422448	948
43a	1	25421536	25424196	-2660
160a	1	25424446	25421488	2958
192a	1	25418843	25426859	-8016
2340i	1	25425512	25419660	5852
10336d	1	25421245	25423628	-2383
389a	2	25427014	25418738	8276
433a	2	25425902	25419896	6006
2432d	2	25423818	25421900	1918
3776h	2	25422350	25422750	-400
5077a	3	25426985	25418831	8154
11197a	3	25429098	25416702	12396

To deal with the more delicate structure of these statistics, consider the following 'Sum of local data'

$$\frac{\log X}{\sqrt{X}} \sum_{p \leq X} \gamma_E(p)$$

where $\gamma_E(p) = 0$ if p is a bad or supersingular prime for E and is otherwise is $+1$ if E has less that $p+1$ rational points over $\mathbf{F}_p$; and $\gamma_E(p) = -1$ if E has more that $p+1$ points. Then this sum, which will be denoted $\Delta_E(X)$, measures exactly the difference between over-count and under-count[6].

The mean of $\Delta_E(X)$ is (conjecturally)

$$\frac{2}{\pi} - \frac{16}{3\pi} r_E \quad + \quad \frac{4}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \Big(\frac{1}{2k+1} + \frac{1}{2k+3} \Big) r_E(2k+1).$$

where

$$r_E(n) := \; r_{f_E}(n) \; = \; \text{the order of vanishing of } L(symm^n f_E, s) \text{ at } s = 1/2,$$

with $f_E :=$ the newform of weight two corresponding to the elliptic curve E; and where we have normalized things so that $s = 1/2$ is the central point. **NOTE:** For a discussion of the numerics of the values $r_E(2k+1)$, see Section **??** below.

[6] Ralph Greenberg has raised the following question: if E and F are elliptic curves such that $\gamma_E(p) = \gamma_F(p)$ for all (or almost all?) p, are E and F necessarily isogenous? TODO: I asked Ralph and he said that there is now a paper by some Koreans about this, with a nice conditional answer, evidently following approach that I suggested.

3.1 The bias between under-counts and over-counts

We will assume that our data has 'Explicit Formula' statistics, and—copying Sarnak ([?])— apply this to the question we began with, i.e., what is the "bias" in the race between under-counts and over-counts?

$$\Delta_E(X) := \frac{\log X}{\sqrt{X}}\big(\#\{p < X \mid N_E(p) < p+1\} \; - \; \#\{p < X \mid N_E(p) > p+1\}\big).$$

Let $H(\theta)$ be the Heaviside function, i.e., the function with value

$$H(\theta) \;=\; +1 \tag{3.1}$$

for $\theta \in [0, \pi/2)$ and -1 for $\theta \in [\pi/2, \pi)$. So

$$\Delta_E(X) = \frac{\log X}{\sqrt{X}} \sum_{p \leq X} H(\theta_p) \tag{3.2}$$

For $n \geq 0$, set

$$c_n(H) \;=\; \langle H, U_n\rangle \;=\; \frac{2}{\pi}\Big[\int_0^{\pi/2} U_n \sin^2\theta d\theta - \int_{\pi/2}^{\pi} U_n \sin^2\theta d\theta\Big] \tag{3.3}$$

which is 0 if n is even and

$$(-1)^{(n-1)/2}\frac{2}{\pi}\Big[\frac{1}{n} + \frac{1}{n+2}\Big]$$

if n is odd.

For $N \geq 1$ let

$$H_N(\theta) := \sum_{n=1}^{N} c_n(H) U_n(\theta) \tag{3.4}$$

So H_N is a smoothed out version of $H(\theta)$ and $H_N(\theta) \to H(\theta)$ as N tends to infinity. Thus

$$S_N(X) := S_{H_N}(X) = \frac{\log X}{\sqrt{X}} \sum_{p \leq X} H_N(\theta_p) \tag{3.5}$$

is a smoothed out version of

$$S(X) := S_H(X) = \frac{\log X}{\sqrt{X}} \sum_{p \le X} H(\theta_p) \tag{3.6}$$

Therefore, by formula (2.6), we would have:

$$\mathcal{E}(S_N) \;=\; \frac{8}{3\pi}(1-2r) + \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\big[\frac{1}{2k+1} + \frac{1}{2k+3}\big]\big(2r_E(2k+1)-1\big). \tag{3.7}$$

Now one does have parity information concerning the arithmetic function $n \mapsto r_E(n)$. For a detailed study of the root numbers of L-functions of symmetric powers of an elliptic curve, consult [**?**]. For $n \ge 1$ let $\nu_E(n) \in \{0,1\}$ be (zero or one) such that $\nu_E(n) \equiv r_E(n)$ modulo 2. Let $s_E(n)$ be the non-negative integer such that:

$$r_E(n) = \nu_E(n) + 2s_E(n)$$

(for $n \ge 3$, odd). Thus if the multiplicity of order of vanishing at the central point $s = 1/2$ of the odd symmetric n-th power L-functions attached to E (for $n \ge 3$) were never greater than 1, and hence entirely dictated by parity, then the conjectured mean, $\mathcal{E}(S_N)$, would be equal to

$$\mathcal{T}_E^{\{N\}} \;:=\; \frac{8}{3\pi}(1-2r_E) + \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\big[\frac{1}{2k+1} + \frac{1}{2k+3}\big]\big(2\nu_E(2k+1)-1\big). \tag{3.8}$$

Now consider the limit:

$$\mathcal{T}_E := \lim_{N\to\infty} \mathcal{T}_E^{\{N\}}.$$

Note: in the semistable case,

$$\mathcal{T}_E = \frac{8 \pm 2}{3\pi} - \frac{16}{3\pi} r_E,$$

where the sign depends on whether $\nu_E(2k+1)$ is 1 or 0.

Put

$$\mathcal{Z}_E^{\{N\}} := \frac{2}{\pi}\sum_{k=1}^{N}(-1)^{k+1}\big[\frac{1}{2k+1} + \frac{1}{2k+3}\big]\big(4s_E(2k+1)\big).$$

Questions: Does the limit,

$$\mathcal{Z}_E := \lim_{N\to\infty} \mathcal{Z}_E^{\{N\}}$$

exist? Does it converge to a finite value? If so, then the conjectured mean would be:

$$\mathcal{E}_E = \mathcal{T}_E \;+\; \mathcal{Z}_E.$$

Is s_{2k+1} bounded? Is the set of positive integers k such that $s_{2k+1} \neq 0$ of *density zero* set of positive integers k? Is that set finite?

Some data for higher order of vanishing for symmetric powers is given in the article of Martin and Watkins [**?**]. The following table is taken from their article:

E	k	s_{2k+1}
$2379b$	1	2
$5423a$	1	2
$10336d$	1	2
$29862s$	1	2
$816b$	2	1
$2340i$	2	1
$2432d$	2	1
$3776h$	2	1
$128b$	3	1
$160a$	3	1
$192a$	3	1

4 Recent work and further questions

- *The relationship between bias and unbounded rank: the work of Fiorilli*

  In the work of Sarnak and Fiorilli, another measure for understanding 'bias behavior' is given by what one might call **the percentage of positive support** (relative to the multiplicative measure dX/X). Namely [[define $\delta(x)$!]]:

$$\mathcal{P} = \mathcal{P}_E := \liminf_{X\to\infty} \frac{1}{\log X} \int_{2\le x\le X;\delta(x)\le 0} dx/x$$

$$= \limsup_{X\to\infty} \frac{1}{\log X} \int_{2\le x\le X;\delta(x)\le 0} dx/x$$

  It is indeed a conjecture, in specific instances interesting to us, that these limits $\mathcal{E}$ and $\mathcal{P}$ exist.

  The standard conjecture (that we have been making all along) is GRH. But here, one includes the further conjecture (given in Sarnak's letter, and the article of Fiorilli) that the the set of nontrivial complex zeroes of the relevant L-function $L(E,s)$ with positive imaginary part is a set of complex numbers that are *linearly independent* over $\mathbf{Q}$. Such a conjecture Rubenstein and Sarnak refer to in [**?**] as the *Grand Simplicity Hypothesis* (GSH). Fiorilli calls his version of it *Hypothesis LI(E)*. For recent, somewhat related, work on such linear independence questions, see [**?**]. Fiorilli, following the work of Sarnak, proves:

Theorem 4.1. *Assume GRH and LI(E). Then the following two statements are equivalent:*

1. *The set of (analytic) ranks $\{r_E\}_E$ ranging over all elliptic curves over* $\mathbf{Q}$ *is it unbounded.*
2. *The l.u.b of the set of percentages of positive support $\{\mathcal{P}_E\}_E$ is equal to* 1.

- *The relationship between bias and bounding the rank: the work of Bober* In [**?**], Jonathan Bober builds on work of Odlyzko and Mestre to establish a conditional upper bound on the ranks of various known elliptic curves of (relatively) high Mordell-Weil rank, notably Noam Elkies' elliptic curve E_{28} for which 28 linearly independent rational points have been found; Bober shows, conditional on the Birch-Swinnerton-Dyer conjecture and GRH, that the Mordell-Weil rank of E_{28} is either 28 or 30 (subsequently, Jamie Weigandt used the same approach to verify that the rank is indeed 28). He does this by a nice 'bias' computation using the Explicit Formula.

- Simon Spicer has further built on Bober's work to create an algorithm for quickly bounding the analytic ranks, which is used on hundreds of millions of elliptic curves in [**?**, §2.3].

to be inserted

- *Further questions*

  In summary, given the conjectures discussed, the *theory of the means* of the weighted sums of local data we have been examining related to a non-CM elliptic curve E is determined by the orders of vanishing at the central point of the L-functions of the symmetric powers of the modular eigenform attached to E: and conversely: knowledge of the means of all such weighted sums determines (conjecturally, of course) all those orders of vanishing; i.e., the arithmetic function $k \mapsto r_E(2k+1)$ (cf. (2.6) above).

  - Is $r_E(2k+1) \geq 2$ for only a set of values of k of density 0?
  - Is $r_E(2k+1) \geq 2$ for all but finitely many k's?
  - Which weighted sums of local data have finite mean?
  - Is there an *effective version* of the conjecture LI(E)? I.e., can we find an explicit positive function $\Phi(H,T)$ such that for every linear combination of the form

    $$\sum_{j=1}^{\nu} \lambda_j \gamma_j$$

    with the λ_j 's rational numbers of height $< H$ and the γ_j 's positive imaginary parts $< T$ of the complex zeroes of the L function $L(E,s)$, we have an inequality of the form

    $$\left| \sum_{j=1}^{\nu} \lambda_j \gamma_j \right| > \Phi(H,T)?$$

    [[idea – instead just define $\Phi(H,T)$ to be the min of the finitely many possible linear combinations, and ask about the behavior of Φ. Is there some clever algorithm to get information about this?]]