Calculations for *Origin of Cosmological Temperature*

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This note shows calculations for the paper *Origin of Cosmological Temperature*. The calculations are elementary. They are written out so that the details can be checked. The numerical calculations (mostly simple arithmetic) are shown in Appendix A. They were performed in a SageMath notebook that is included in the Supplemental Materials.

The Supplemental Materials — the SageMath notebook, renditions of it into html and pdf, and this note — are available at `https://cocalc.com/share/3c1ab84c375769c3460251d4f2bd43461c1211b5/Supplemental_material/` and at `http://www.physics.rutgers.edu/pages/friedan/papers/2020/Origin/Supplemental_material/`.

To run the notebook, either install SageMath from `sagemath.org` or create an account at `cocalc.com` and upload the notebooks to run there. SageMath is a free open-source mathematics software system. Both free and paid accounts are available at `cocalc.com`. Paying for an account supports SageMath (see `reasons-for-purchasing-a-subscription`). Open-source mathematics software such as SageMath is essential for scientific research. Scientific results must be open to scrutiny. Closed-source mathematics software blocks scrutiny.

Contents

1 Classical action

Consider the classical equations of motion of the Standard Model combined with General Relativity, assuming

1. the only nonzero fields are
   - the space-time metric $g_{\mu\nu}(x)$,
   - the SU(2) gauge field $B_\mu(x) \in \mathfrak{su}(2)$,
   - the Higgs scalar field $\phi(x) \in \mathbb{C}^2$,
2. space is the 3-sphere S^3, and
3. the universe is Spin(4)-symmetric.

The action (in units of $\hbar$ with $c = 1$) is

$$\begin{aligned} \frac{1}{\hbar}S &= \frac{1}{\hbar}S_{\mathrm{grav}} + \frac{1}{\hbar}S_{\mathrm{gauge}} + \frac{1}{\hbar}S_{\mathrm{scalar}} \\ \frac{1}{\hbar}S_{\mathrm{grav}} &= \int \frac{1}{\hbar}\frac{R}{2\kappa}\sqrt{-g}\, d^4x \\ \frac{1}{\hbar}S_{\mathrm{gauge}} &= \int \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\sqrt{-g}\, d^4x \\ \frac{1}{\hbar}S_{\mathrm{scalar}} &= \int \left[-D_\mu\phi^\dagger D^\mu\phi - \frac{1}{2}\lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2\right]\sqrt{-g}\, d^4x \end{aligned} \tag{1.1}$$

The gauge covariant derivative and curvature 2-form are

$$D_\mu\phi = (\partial_\mu + B_\mu)\phi \qquad F_{\mu\nu} = [D_\mu,\, D_\nu] = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu,\, B_\nu] \tag{1.2}$$

2 Spin(4) symmetry

Space is the 3-sphere S^3. Space-time is $I \times S^3$ where time I is a real interval. Use indices μ, ν for points x^μ in space-time. Parametrize space-time by $x = (x^0, \hat{x})$ with $x^0 \in I$ and $\hat{x}$ in the unit 3-sphere S^3 in euclidean 4-space. Use indices j, k for euclidean 4-space.

$$\hat{x} \in S^3 \subset \mathbb{R}^4 \qquad \hat{x} = (\hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{x}^4) \qquad \hat{x}_k\hat{x}^k = 1 \qquad \hat{x}_k = \hat{x}^j\delta_{jk} \qquad \hat{x}_k d\hat{x}^k = 0 \tag{2.1}$$

$\mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is the simply connected covering group of SO(4). Let σ_a, $a = 1, 2, 3$ be the Pauli matrices.

$$\sigma_a^\dagger = \sigma_a \qquad \sigma_a\sigma_b = \delta_{ab}\mathbf{1} + \epsilon_{abc}i\sigma_c \qquad \mathrm{tr}(\sigma_a\sigma_b) = 2\delta_{ab} \tag{2.2}$$

The anti-hermitian matrices $i^{-1}\sigma_a$ form a basis for the Lie algebra $\mathfrak{su}(2)$. Use indices a, b, c for elements of $\mathfrak{su}(2)$.

2.1 Identify S^3 with SU(2)

Identify the unit 3-sphere S^3 with the group SU(2) by

$$\hat{x} \quad \longleftrightarrow \quad g_{\hat{x}} = \hat{x}^4 \mathbf{1} + \hat{x}^a i^{-1} \sigma_a = \begin{pmatrix} \hat{x}^4 - i\hat{x}^3 & -i\hat{x}^1 - \hat{x}^2 \\ -i\hat{x}^1 + \hat{x}^2 & \hat{x}^4 + i\hat{x}^3 \end{pmatrix} \tag{2.3}$$

checking that

$$\begin{aligned} (g_{\hat{x}})^\dagger g_{\hat{x}} &= (\hat{x}^4 \mathbf{1} - \hat{x}^a i^{-1}\sigma_a)(\hat{x}^4 \mathbf{1} + \hat{x}^b i^{-1}\sigma_b) = (\hat{x}^4)^2\, \mathbf{1} + \hat{x}^a \hat{x}^b \sigma_a \sigma_b = \delta_{jk}\hat{x}^j\hat{x}^k\, \mathbf{1} = \mathbf{1} \\ \det g_{\hat{x}} &= |\hat{x}^4 - i\hat{x}^3|^2 + |-i\hat{x}^1 - \hat{x}^2|^2 = \delta_{jk}\hat{x}^j\hat{x}^k = 1 \end{aligned} \tag{2.4}$$

The north-pole $\hat{N}$ in S^3 is identified with the identity element $\mathbf{1}$ in SU(2)

$$\hat{N} = (0,0,0,1) \qquad g_{\hat{N}} = \mathbf{1} \tag{2.5}$$

The tangent space to S^3 at $\hat{N}$ is identified with the Lie algebra $\mathfrak{su}(2)$

$$dg_{\hat{x}}(\hat{N}) = d\hat{x}^a\, i^{-1}\sigma_a \tag{2.6}$$

The identity

$$\operatorname{tr}(g_{\hat{x}}^{-1} g_{\hat{y}}) = \operatorname{tr}(g_{\hat{x}}^\dagger g_{\hat{y}}) = \operatorname{tr}\Big((\hat{x}^4)^2\, \mathbf{1} + \hat{x}^a \hat{x}^b \sigma_a \sigma_b \Big) = 2\delta_{jk}\hat{x}^j \hat{y}^k \tag{2.7}$$

identifies the metric of the unit 3-sphere with the Killing form on SU(2)

$$ds^2_{S^3} = \delta_{jk} d\hat{x}^j d\hat{y}^k = \frac{1}{2}\operatorname{tr}(dg_{\hat{x}}^{-1} dg_{\hat{y}}) \tag{2.8}$$

At the north-pole the metric of the unit 3-sphere is

$$ds^2_{S^3}(\hat{N}) = \delta_{ab} d\hat{x}^a d\hat{x}^b \tag{2.9}$$

The metric volume element of the unit 3-sphere is

$$d^3\hat{x} = d^4x\, \delta(r-1) \qquad r^2 = x_k x^k \qquad \int_{S^3} d^3\hat{x} = 2\pi^2 \tag{2.10}$$

2.2 Spin(4) acts as SO(4) on space-time

An element $U = (g_L, g_R)$ in $\mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ acts on S^3 by

$$U = (g_L, g_R) \qquad g_{\hat{x}} \mapsto g_L g_{\hat{x}} g_R^{-1} \qquad \hat{x} \mapsto U\hat{x} \qquad g_{U\hat{x}} = g_L g_{\hat{x}} g_R^{-1} \tag{2.11}$$

$\hat{x} \mapsto U\hat{x}$ is an SO(4) rotation because the identity (2.7) implies

$$\delta_{jk}(U\hat{x})^j (U\hat{y})^k = \delta_{jk}\hat{x}^j\hat{y}^k \tag{2.12}$$

The map Spin(4) $\to$ SO(4) is 2-to-1 because $(g_L, g_R) = (-1,-1)$ acts as the identity. Spin(4) acts on space-time by rotating space independent of time

$$x \mapsto Ux \qquad (x^0, \hat{x}) \mapsto (x^0, U\hat{x}) \tag{2.13}$$

Spin(4) takes the north-pole $\hat{N}$ to any other point in S^3 so a Spin(4)-symmetric field is completely determined by its value at $\hat{N}$. The value at $\hat{N}$ must be invariant under the little group at $\hat{N}$, the subgroup that takes $\hat{N}$ to itself, $g_L \mathbf{1} g_R^{-1} = \mathbf{1}$, which is the diagonal subgroup $\mathrm{SU}(2)_\Delta = \{(g_L, g_L)\}$. The little group acts on the tangent space to S^3 at $\hat{N}$ as the adjoint action of SU(2) on the Lie algebra $\mathfrak{su}(2)$.

$$\text{at } \hat{x} = \hat{N}, \qquad d\left(g_L g_{\hat{x}} g_L^{-1}\right) = d\hat{x}^a\, g_L \left(i^{-1}\sigma_a\right) g_L^{-1} \tag{2.14}$$

2.3 SO(4)-symmetric metric

If $g_{\mu\nu}(x)$ is an SO(4)-invariant space-time metric, its value at the north-pole is

$$\begin{aligned} g_{\mu\nu}(\hat{N})dx^\mu dx^\nu &= g_{00}(x^0)(dx^0)^2 + g_{0a}(x^0)(dx^0 d\hat{x}^a + d\hat{x}^a dx^0) + g_{ab}(x^0)d\hat{x}^a d\hat{x}^b \\ &= -F_1(x^0)^2(dx^0)^2 + F_2(x^0)^2\delta_{ab}d\hat{x}^a d\hat{x}^b \end{aligned} \tag{2.15}$$

The off-diagonal term $g_{0a}(x^0) = 0$ because it must be an element of $\mathfrak{su}(2)$ invariant under the adjoint action. The only such element is 0. The spatial metric $g_{ab}(x^0)$ is an invariant positive bilinear form on $\mathfrak{su}(2)$ therefore a multiple of the Killing form. So the general SO(4)-invariant space-time metric has the form

$$g_{\mu\nu}(x)dx^\mu dx^\nu = -F_1(x^0)^2(dx^0)^2 + F_2(x^0)^2 ds^2_{S^3} \tag{2.16}$$

where $ds^2_{S^3}$ is the metric of the unit 3-sphere.

Make a reparametrization of time $x^0 \to T(x^0)$ such that

$$\frac{dT}{dx^0} = \frac{F_1(x^0)}{F_2(x^0)} \tag{2.17}$$

and define $a(T) = F_2(x^0)$. The space-time metric is now in the conformally flat form

$$g_{\mu\nu}(x)dx^\mu dx^\nu = a(T)^2\left(-dT^2 + ds^2_{S^3}\right) \tag{2.18}$$

To see that this metric is conformally flat, Wick rotate to imaginary time $\tau = iT$. Then

$$\begin{aligned} g_{\mu\nu}(x)dx^\mu dx^\nu &= a^2\left(d\tau^2 + ds^2_{S^3}\right) \\ &= a^2 r^{-2}\left(dr^2 + r^2 ds^2_{S^3}\right) \qquad r = e^\tau \\ &= a^2 r^{-2}\delta_{jk}dx^j dx^k \qquad r^2 = x_k x^k \end{aligned} \tag{2.19}$$

so $g_{\mu\nu}(x)$ is conformal to the flat euclidean metric on $\mathbb{R}^4$ with conformal factor a^2r^{-2}.

The metric volume element is

$$\sqrt{-g}\, d^4x = a^4 dT\, d^3\hat{x} \tag{2.20}$$

where $d^3\hat{x}$ is the volume element of the unit 3-sphere.

2.4 SU(2) scalar and gauge fields

All SU(2) vector bundles over S^3 are trivial bundles, so the fundamental SU(2) vector bundle on space-time can be identified with the product space $(I \times S^3) \times \mathbb{C}^2$. An SU(2) scalar field $\phi(x)$ is a function on space-time whose value at each point is a complex 2-vector

$$\phi(x) = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}(x) \tag{2.21}$$

i.e., a section of the fundamental SU(2) vector bundle.

An SU(2) gauge field is an $\mathfrak{su}(2)$-valued 1-form $B(x)$ on space-time. The gauge covariant derivative is

$$\begin{aligned} D = d + B \qquad D = dx^\mu D_\mu \qquad D_\mu = \partial_\mu + B_\mu(x) \\ B = dx^\mu B_\mu(x) \qquad B_\mu(x) = B^a_\mu(x)\, i^{-1}\sigma_a \qquad D_\mu\phi(x) = \partial_\mu\phi(x) + B_\mu(x)\phi(x) \end{aligned} \tag{2.22}$$

The curvature 2-form is

$$\begin{aligned} F = D \wedge D = dB + B \wedge B \\ F = \frac{1}{2}dx^\mu \wedge dx^\nu F_{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu,\, B_\nu] \end{aligned} \tag{2.23}$$

2.5 Spin(4)-symmetric scalar and gauge fields

Let $\mathrm{Spin}(4) = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ act on the fundamental vector bundle by

$$x = (T, g_{\hat{x}}) \qquad U = (g_L, g_R) \qquad \phi(x) \mapsto g_L^{-1}\,\phi(Ux) = g_L^{-1}\,\phi(T, g_L g_{\hat{x}} g_R^{-1}) \tag{2.24}$$

so Spin(4) acts on gauge fields by

$$B(x) \mapsto g_L^{-1}\,B(Ux)\,g_L \qquad dx^\mu B_\mu(x) \mapsto d(Ux)^\mu g_L^{-1}\,B_\mu(Ux)\,g_L \tag{2.25}$$

Spin(4)-symmetric fields are left unchanged

$$\phi(x) = g_L^{-1}\,\phi(Ux) \qquad B(x) = g_L^{-1}\,B(Ux)\,g_L \tag{2.26}$$

For an invariant scalar field $\phi(T, \hat{x})$, the value at $\hat{x} = \hat{N}$ is a vector $\phi(T, \hat{N}) \in \mathbb{C}^2$. An element (g_L, g_L) of the little group takes $\hat{N}$ to itself so

$$\phi(T, \hat{N}) \to g_L^{-1}\phi(T, \hat{N}) \tag{2.27}$$

Invariance under the little group means that $\phi(T, \hat{N})$ is invariant under every $g_L \in \mathrm{SU}(2)$. The only such vector is 0. So $\phi(x) = 0$.

For an invariant gauge field $B(x)$, the value at $\hat{N}$ is a 1-form

$$B(T, \hat{N}) = dT B_0^a(T, \hat{N}) i^{-1}\sigma_a + d\hat{x}^b B_b^a(T, \hat{N}) i^{-1}\sigma_a \tag{2.28}$$

Invariance under the little group requires B_0^a to be an element of $\mathfrak{su}(2)$ invariant under the adjoint SU(2) action. The only invariant element of $\mathfrak{su}(2)$ is 0. So $B_0^a = 0$. B_b^a must be an invariant linear map from $\mathfrak{su}(2)$ to itself so must be proportional to the identity. Write this proportionality

$$B_b^a(T, \hat{N}) = [1 + b(T)] \left(-\frac{1}{2}\delta_b^a\right) \tag{2.29}$$

Define the $\mathfrak{su}(2)$-valued 1-form on S^3

$$\gamma(\hat{x}) = -\frac{1}{2}\,(dg_{\hat{x}})\,g_{\hat{x}}^{-1} = \frac{1}{2} g_{\hat{x}} dg_{\hat{x}}^\dagger \tag{2.30}$$

$\gamma(\hat{x})$ is Spin(4)-symmetric.

$$g_L^{-1}\gamma(U\hat{x})g_L = -g_L^{-1}\frac{1}{2}d\left(g_L g_{\hat{x}} g_R^{-1}\right)\left(g_L g_{\hat{x}} g_R^{-1}\right)^{-1} g_L = \gamma(\hat{x}) \tag{2.31}$$

Its value at the north-pole is a multiple of the identity matrix.

$$\gamma(\hat{N}) = -\frac{1}{2}d\hat{x}^a i^{-1}\sigma_a = d\hat{x}^b\left(-\frac{1}{2}\delta_b^a\right) i^{-1}\sigma_a \tag{2.32}$$

so at the north-pole

$$B(T, \hat{N}) = [1 + b(T)]\,\gamma(\hat{N}) \tag{2.33}$$

so by Spin(4)-symmetry the identity holds everywhere

$$B(T, \hat{x}) = [1 + b(T)]\,\gamma(\hat{x}) \tag{2.34}$$

3 Classical action of the Spin(4)-symmetric fields

3.1 Ricci tensor

For the conformally flat space-time metric (2.18)

$$g_{\mu\nu}(x)dx^\mu dx^\nu = a(T)^2\left(-dT^2 + ds^2_{S^3}\right) \tag{3.1}$$

now calculate the Ricci tensor $R_{\mu\nu}$. The result will be

$$R_{\mu\nu}dx^\mu dx^\nu = \left[-3a^{-1}\partial_T^2 a + 3a^{-2}(\partial_T a)^2\right]dT^2 + \left[a^{-1}\partial_T^2 a + a^{-2}(\partial_T a)^2 + 2\right]ds^2_{S^3} \tag{3.2}$$

from which the scalar curvature is

$$\begin{aligned} R = g^{\mu\nu}R_{\mu\nu} &= -a^{-2}\left[-3a^{-1}\partial_T^2 a + 3a^{-2}(\partial_T a)^2\right] + 3a^{-2}\left[a^{-1}\partial_T^2 a + a^{-2}(\partial_T a)^2 + 2\right] \\ &= 6a^{-2}\left(a^{-1}\partial_T^2 a + 1\right) \end{aligned} \tag{3.3}$$

and the Einstein tensor is

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\ G_{\mu\nu}dx^\mu dx^\nu &= \left[-3a^{-1}\partial_T^2 a + 3a^{-2}(\partial_T a)^2\right]dT^2 + \left[a^{-1}\partial_T^2 a + a^{-2}(\partial_T a)^2 + 2\right]ds^2_{S^3} \\ &\quad - \frac{1}{2}6a^{-2}\left(a^{-1}\partial_T^2 a + 1\right)a^2\left(-dT^2 + ds^2_{S^3}\right) \\ &= \left[3a^{-2}(\partial_T a)^2 + 3\right]dT^2 + \left[a^{-2}(\partial_T a)^2 - 2a^{-1}\partial_T^2 a - 1\right]ds^2_{S^3} \end{aligned} \tag{3.4}$$

To calculate the Ricci tensor, first analytically continue to imaginary time $\tau = iT$,

$$g_{\mu\nu}(x)dx^\mu dx^\nu = a^2\left(d\tau^2 + ds^2_{S^3}\right) \qquad \tau = iT \tag{3.5}$$

then change space-time coordinates from $(\tau, \hat{x}^k)$ to $x^k = r\hat{x}^k$, $r = e^\tau$,

$$\begin{aligned} x^k = r\hat{x}^k \qquad r = e^\tau \\ g_{\mu\nu}(x)dx^\mu dx^\nu = g_{jk}dx^j dx^k = a^2 r^{-2}(dr^2 + r^2 ds^2_{S^3}) = a^2 r^{-2}\delta_{jk}dx^j dx^k \\ g_{jk} = e^{2f}\delta_{jk} \qquad e^{2f} = a^2 r^{-2} \end{aligned} \tag{3.6}$$

The metric covariant derivative of a vector field $v^i(x)$ is

$$\begin{aligned} \nabla_k v^i = \partial_k v^i + \Gamma^i_{jk}v^j \\ \Gamma^i_{jk} = \frac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{jm} - \partial^i g_{jk}) = \partial_j f\delta^i_k + \partial_k f\delta^i_j - \partial^i f\delta_{jk} \end{aligned} \tag{3.7}$$

using δ_{ij} to raise and lower indices. The metric curvature tensor is

$$\begin{aligned} (\nabla_j\nabla_k - \nabla_k\nabla_j)v^m = R^m_{ijk}v^i \\ R^m_{ijk} = \partial_j\Gamma^m_{ki} - \partial_k\Gamma^m_{ji} + \Gamma^m_{nj}\Gamma^n_{ik} - \Gamma^m_{nk}\Gamma^n_{ij} \end{aligned} \tag{3.8}$$

The Ricci tensor is

$$R_{ik} = R^j_{ijk} = \partial_j\Gamma^j_{ki} - \partial_k\Gamma^j_{ji} + \Gamma^j_{nj}\Gamma^n_{ik} - \Gamma^j_{nk}\Gamma^n_{ij} \tag{3.9}$$

Substituting,

$$\begin{aligned} R_{ik} &= \partial_j(\partial_k f\delta_i^j + \partial_i f\delta_k^j - \partial^j f\delta_{ki}) - \partial_k(\partial_j f\delta_i^j + \partial_i f\delta_j^j - \partial^j f\delta_{ji}) \\ &\quad + (\partial_n f\delta_j^j + \partial_j f\delta_n^j - \partial^j f\delta_{nj})(\partial_i f\delta_k^n + \partial_k f\delta_i^n - \partial^n \delta_{ik}) \\ &\quad - (\partial_n f\delta_k^j + \partial_k f\delta_n^j - \partial^j f\delta_{nk})(\partial_i f\delta_j^n + \partial_j f\delta_i^n - \partial^n f\delta_{ij}) \\ &= -2\partial_i\partial_k f - \partial^j\partial_j f\delta_{ik} + 2\partial_i f\partial_k f - 2\partial_j f\partial^j f\delta_{ik} \end{aligned} \tag{3.10}$$

Now use that f depends only on $r = e^\tau$

$$\begin{gathered} \partial_i\tau = r^{-1}\hat{x}_i \qquad \partial_i\partial_k\tau = r^{-2}(\delta_{ik} - 2\hat{x}_i\hat{x}_k) \\ \partial_i f = \partial_i\tau\partial_\tau f = r^{-1}\hat{x}_i\partial_\tau f \\ \partial_i\partial_k f = \partial_i\partial_k\tau\partial_\tau f + \partial_i\tau\partial_k\tau\partial_\tau^2 f = r^{-2}\left[(\delta_{ik} - 2\hat{x}_i\hat{x}_k)\partial_\tau f + \hat{x}_i\hat{x}_k\partial_\tau^2 f\right] \\ \partial_j\partial^j f = r^{-2}\left(2\partial_\tau f + \partial_\tau^2 f\right) \end{gathered} \tag{3.11}$$

$$\begin{aligned} R_{ik} &= -2r^{-2}\left[(\delta_{ik} - 2\hat{x}_i\hat{x}_k)\partial_\tau f + \hat{x}_i\hat{x}_k\partial_\tau^2 f\right] - r^{-2}\left(2\partial_\tau f + \partial_\tau^2 f\right)\delta_{ik} \\ &\quad + 2r^{-2}\hat{x}_i\hat{x}_k(\partial_\tau f)^2 - 2r^{-2}(\partial_\tau f)^2\delta_{ik} \\ &= r^{-2}\hat{x}_i\hat{x}_k\left[4\partial_\tau f - 2\partial_\tau^2 f + 2(\partial_\tau f)^2\right] + r^{-2}\delta_{ik}\left[-4\partial_\tau f - \partial_\tau^2 f - 2(\partial_\tau f)^2\right] \\ &= r^{-2}\hat{x}_i\hat{x}_k\left(-3\partial_\tau^2 f\right) + r^{-2}(\delta_{km} - \hat{x}_i\hat{x}_k)\left[-4\partial_\tau f - \partial_\tau^2 f - 2(\partial_\tau f)^2\right] \end{aligned} \tag{3.12}$$

Substitute

$$f = \ln a - \tau \qquad \partial_\tau f = a^{-1}\partial_\tau a - 1 \qquad \partial_\tau^2 f = a^{-1}\partial_\tau^2 a - a^{-2}(\partial_\tau a)^2 \tag{3.13}$$

to get

$$\begin{aligned} R_{ik} &= r^{-2}\hat{x}_i\hat{x}_k\left[-3a^{-1}\partial_\tau^2 a + 3a^{-2}(\partial_\tau a)^2\right] \\ &\quad + r^{-2}(\delta_{ik} - \hat{x}_i\hat{x}_k)\left[-4a^{-1}\partial_\tau a + 4 - a^{-1}\partial_\tau^2 a + a^{-2}(\partial_\tau a)^2\right. \\ &\qquad\qquad \left. - 2(a^{-1}\partial_\tau a - 1)^2\right] \\ &= r^{-2}\hat{x}_i\hat{x}_k\left[-3a^{-1}\partial_\tau^2 a + 3a^{-2}(\partial_\tau a)^2\right] \\ &\quad + r^{-2}(\delta_{ik} - \hat{x}_i\hat{x}_k)\left[2 - a^{-1}\partial_\tau^2 a - a^{-2}(\partial_\tau a)^2\right] \end{aligned} \tag{3.14}$$

Finally return to real time T,

$$r^{-2}\hat{x}_i\hat{x}_k dx^i dx^k = d\tau^2 = -dT^2 \qquad r^{-2}(\delta_{ik} - \hat{x}_i\hat{x}_k)dx^i dx^k = ds_{S^3}^2 \tag{3.15}$$

to get the Ricci tensor formula

$$\begin{aligned} R_{\mu\nu}dx^\mu dx^\nu &= R_{ik}dx^i dx^k \\ &= \left[-3a^{-1}\partial_T^2 a + 3a^{-2}(\partial_T a)^2\right]dT^2 + \left[a^{-1}\partial_T^2 a + a^{-2}(\partial_T a)^2 + 2\right]ds_{S^3}^2 \end{aligned} \tag{3.16}$$

3.2 Gravitational action

Using formula (3.3) for the scalar curvature, the gravitational action is

$$\begin{aligned} \frac{1}{\hbar}S_{\text{grav}} &= \int \frac{1}{2\kappa\hbar}R\sqrt{-g}\,d^4x \\ &= \int \frac{1}{2\kappa\hbar}6a^{-2}\left(a^{-1}\partial_T^2 a + 1\right)\,a^4 dT \int_{S^3} d^3\hat{x} \\ &= \frac{12\pi^2}{\kappa\hbar}\int \frac{1}{2}\left(a\partial_T^2 a + a^2\right)\,dT \end{aligned} \tag{3.17}$$

Integrating by parts and discarding the boundary terms gives

$$\frac{1}{\hbar}S_{\text{grav}} = \frac{12\pi^2}{\kappa\hbar}\int \frac{1}{2}\left[-(\partial_T a)^2 + a^2\right]\, dT \tag{3.18}$$

3.3 Gauge field action

To find the gauge field action of the Spin(4)-symmetric gauge field (2.34)

$$\frac{1}{\hbar}S_{\text{gauge}} = \int \frac{1}{2g^2}\text{tr}(F_{\mu\nu}F^{\mu\nu})\sqrt{-g}\, d^4x \qquad B(T,\hat{x}) = [1 + b(T)]\,\gamma(\hat{x}) \tag{3.19}$$

in the SO(4)-symmetric conformally flat metric (2.18)

$$g_{\mu\nu}(x)dx^\mu dx^\nu = a(T)^2\left(-dT^2 + ds^2_{S^3}\right) \tag{3.20}$$

first calculate the curvature 2-form

$$\begin{gathered} dB = (\partial_T b)dT\wedge\gamma + (1+b)d\gamma \\ F = dB + B\wedge B = (\partial_T b)dT\wedge\gamma + (1+b)d\gamma + (1+b)^2\gamma\wedge\gamma \end{gathered} \tag{3.21}$$

then use the identity

$$\begin{aligned} d\gamma &= d\left[-\frac{1}{2}\left(dg_{\hat{x}}\right)g_{\hat{x}}^{-1}\right] = \frac{1}{2}\left(dg_{\hat{x}}\right)\wedge d\left(g_{\hat{x}}^{-1}\right) = -\frac{1}{2}\left(dg_{\hat{x}}\right)\wedge g_{\hat{x}}^{-1}\left(dg_{\hat{x}}\right)g_{\hat{x}}^{-1} \\ &= -2\gamma\wedge\gamma \end{aligned} \tag{3.22}$$

to get

$$F = (\partial_T b)dT\wedge\gamma + (b^2-1)\gamma\wedge\gamma \tag{3.23}$$

Next find the components $F_{\mu\nu}(T,\hat{N})$ at the north-pole.

$$\begin{aligned} F(T,\hat{N}) &= \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu = F_{0a}dT\wedge d\hat{x}^a + \frac{1}{2}F_{ab}d\hat{x}^a\wedge d\hat{x}^b \\ \gamma\wedge\gamma(\hat{N}) &= \left(-\frac{1}{2}d\hat{x}^a i^{-1}\sigma_a\right)\wedge\left(-\frac{1}{2}d\hat{x}^b i^{-1}\sigma_b\right) = \frac{1}{4}d\hat{x}^a\wedge d\hat{x}^b\epsilon_{abc}i^{-1}\sigma_c \\ F(T,\hat{N}) &= (\partial_T b)dT\wedge\left(-\frac{1}{2}d\hat{x}^a i^{-1}\sigma_a\right) + (b^2-1)\frac{1}{4}d\hat{x}^a\wedge d\hat{x}^b\epsilon_{abc}i^{-1}\sigma_c \\ F_{0a}(T,\hat{N}) &= -\frac{1}{2}\partial_T b\, i^{-1}\sigma_a \qquad F_{ab}(T,\hat{N}) = \frac{1}{2}(b^2-1)\epsilon_{abc}\, i^{-1}\sigma_c \end{aligned} \tag{3.24}$$

Then evaluate $\text{tr}\,(F_{\mu\nu}F^{\mu\nu})$ at the north-pole.

$$\begin{aligned} \text{tr}\,(F_{\mu\nu}F^{\mu\nu})\,(T,\hat{N}) &= a^{-4}\text{tr}\left(-2F_{0a}F_{0a'}\delta^{aa'} + F_{ab}F_{a'b'}\delta^{aa'}\delta^{bb'}\right) \\ &= a^{-4}\text{tr}\left(\frac{1}{2}(\partial_T b)^2\delta^{aa'}\sigma_a\sigma_{a'} - \frac{1}{4}(b^2-1)^2\epsilon_{abc}\epsilon_{a'b'c'}\delta^{aa'}\delta^{bb'}\sigma_c\sigma_{c'}\right) \\ &= 3a^{-4}\left[(\partial_T b)^2 - (b^2-1)^2\right] \end{aligned} \tag{3.25}$$

This holds everywhere in space-time because $\text{tr}\,(F_{\mu\nu}F^{\mu\nu})$ is a Spin(4)-invariant function.

$$\text{tr}\,(F_{\mu\nu}F^{\mu\nu}) = 3a^{-4}\left[(\partial_T b)^2 - (b^2-1)^2\right] \tag{3.26}$$

So the gauge field action is

$$
\begin{aligned}
\frac{1}{\hbar} S_{\text{gauge}} &= \int \frac{1}{2g^2} \text{tr}(F_{\mu\nu} F^{\mu\nu}) \sqrt{-g}\, d^4x \\
&= \int \frac{1}{2g^2} 3a^{-4} \left[(\partial_T b)^2 - (b^2-1)^2 \right] a^4 dT \int_{S^3} d^3\hat{x} \\
&= \frac{6\pi^2}{g^2} \int \frac{1}{2} \left[(\partial_T b)^2 - (b^2-1)^2 \right] dT
\end{aligned}
\tag{3.27}
$$

3.4 Scalar field action

The action of the Spin(4)-symmetric scalar field $\phi = 0$ is

$$
\begin{aligned}
\frac{1}{\hbar} S_{\text{scalar}} &= \int \left[-D_\mu \phi^\dagger D^\mu \phi - \frac{1}{2}\lambda^2 \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right)^2 \right] \sqrt{-g}\, d^4x \\
&= \int \left(-\frac{1}{8} \lambda^2 v^4 \right) a^4 dT \int_{S^3} d^3\hat{x} \\
&= 2\pi^2 \int \left(-\frac{1}{8} \lambda^2 v^4 \right) a^4 dT \\
&= 2\pi^2 \int \left(-\frac{1}{\hbar} \mathcal{E}_0 \right) a^4 dT \qquad \frac{1}{\hbar}\mathcal{E}_0 = \frac{1}{8}\lambda^2 v^4
\end{aligned}
\tag{3.28}
$$

3.5 Action in dimensionless variables

The sum of the gravitational action (3.18) and the scalar action (3.28) is

$$
\begin{aligned}
\frac{1}{\hbar} S_{\text{grav}} + \frac{1}{\hbar} S_{\text{scalar}} &= \frac{12\pi^2}{\kappa\hbar} \int \frac{1}{2} \left[-(\partial_T a)^2 + a^2 \right] dT + 2\pi^2 \int \left(-\frac{1}{\hbar}\mathcal{E}_0 \right) a^4 dT \\
&= \frac{12\pi^2}{\kappa\hbar} \int \frac{1}{2} \left[-(\partial_T a)^2 + a^2 - \frac{\kappa \mathcal{E}_0}{3} a^4 \right] dT
\end{aligned}
\tag{3.29}
$$

where

$$
\frac{1}{\hbar}\mathcal{E}_0 = \frac{1}{8}\lambda^2 v^4 \tag{3.30}
$$

Change variable

$$
a = t_{\text{I}} \hat{a} \qquad \frac{1}{\hbar} S_{\text{grav}} + \frac{1}{\hbar} S_{\text{scalar}} = \frac{12\pi^2 t_{\text{I}}^2}{\kappa\hbar} \int \frac{1}{2} \left[-(\partial_T \hat{a})^2 + \hat{a}^2 - \frac{\kappa \mathcal{E}_0 t_{\text{I}}^2}{3} \hat{a}^4 \right] dT \tag{3.31}
$$

Choose t_{I} so that

$$
\frac{\kappa \mathcal{E}_0 t_{\text{I}}^2}{3} = 1 \qquad t_{\text{I}}^2 = \frac{3}{\kappa \mathcal{E}_0} = \frac{24}{\kappa\hbar\lambda^2 v^4} \tag{3.32}
$$

Then

$$
\frac{1}{\hbar} S_{\text{grav}} + \frac{1}{\hbar} S_{\text{scalar}} = -\frac{12\pi^2 t_{\text{I}}^2}{\kappa\hbar} \int \left[\frac{1}{2} (\partial_T \hat{a})^2 - \frac{1}{2} \left(\hat{a}^2 - \hat{a}^4 \right) \right] dT \tag{3.33}
$$

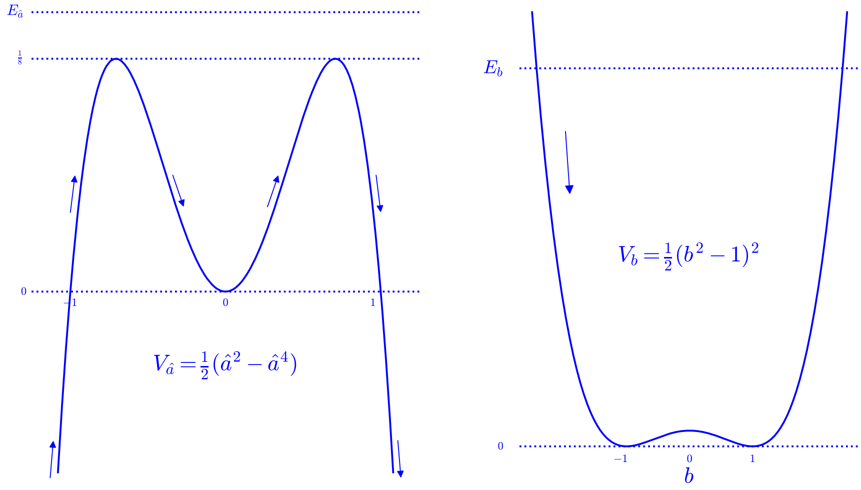


Figure 1: The anharmonic potentials. The energies $E_{\hat{a}}$ and $E_{\hat{b}}$ are not to scale.

Adding the gauge action (3.27), the total action is

$$\begin{aligned}
\frac{1}{\hbar}S &= \frac{1}{\hbar}S_{\text{grav}} + \frac{1}{\hbar}S_{\text{scalar}} + \frac{1}{\hbar}S_{\text{gauge}} \\
&= -\frac{12\pi^2 t_{\text{I}}^2}{\kappa\hbar}\int \left[\frac{1}{2}(\partial_T \hat{a})^2 - \frac{1}{2}\left(\hat{a}^2 - \hat{a}^4\right)\right] dT \\
&\quad + \frac{6\pi^2}{g^2}\int \left[\frac{1}{2}(\partial_T b)^2 - \frac{1}{2}(b^2 - 1)^2\right] dT \\
&= \frac{6\pi^2}{g^2}\int \left(-\frac{g^2}{6\pi^2}\frac{12\pi^2 t_{\text{I}}^2}{\kappa\hbar}\left[\frac{1}{2}(\partial_T \hat{a})^2 - \frac{1}{2}\left(\hat{a}^2 - \hat{a}^4\right)\right]\right. \\
&\qquad\qquad \left. + \left[\frac{1}{2}(\partial_T b)^2 - \frac{1}{2}(b^2 - 1)^2\right]\right) dT
\end{aligned} \tag{3.34}$$

which is

$$\begin{gathered}
\frac{1}{\hbar}S = \frac{6\pi^2}{g^2}\int \left(-\epsilon_{\text{w}}^{-4}\left[\frac{1}{2}(\partial_T \hat{a})^2 - V_{\hat{a}}(\hat{a})\right] + \left[\frac{1}{2}(\partial_T b)^2 - V_b(b)\right]\right) dT \\
\epsilon_{\text{w}}^{-4} = \frac{2g^2 t_{\text{I}}^2}{\kappa\hbar} \qquad V_{\hat{a}}(\hat{a}) = \frac{1}{2}\left(\hat{a}^2 - \hat{a}^4\right) \qquad V_b(b) = \frac{1}{2}(b^2 - 1)^2
\end{gathered} \tag{3.35}$$

The potentials are graphed in Figure 1.

4 Classical equations of motion

4.1 Gravitational equation of motion and energy-momentum tensors

Vary the action (1.1) with respect to a variation of the metric $\delta g_{\mu\nu} = h_{\mu\nu}$.

$$\delta S_{\text{grav}} = \int h^{\mu\nu} \frac{1}{2\kappa}(-G_{\mu\nu}) \sqrt{-g}\, d^4x \tag{4.1}$$

$$= \int h^{\mu\nu} \frac{1}{2\kappa}(-R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu}) \sqrt{-g}\, d^4x$$

$$\delta S_{\text{gauge}} = \int h^{\mu\nu} \frac{1}{2} T^{\text{gauge}}_{\mu\nu} \sqrt{-g}\, d^4x \tag{4.2}$$

$$= \int h^{\mu\nu} \frac{\hbar}{2g^2} \text{tr} \left(-2F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \sqrt{-g}\, d^4x$$

$$\delta S_{\text{scalar}} = \int h^{\mu\nu} \frac{1}{2} T^{\text{scalar}}_{\mu\nu} \sqrt{-g}\, d^4x \tag{4.3}$$

$$= \int h^{\mu\nu}\, \hbar \left[D_\mu \phi^\dagger D_\nu \phi - \frac{1}{2} g_{\mu\nu} D_\sigma \phi^\dagger D^\sigma \phi - \frac{1}{2} g_{\mu\nu} \frac{1}{2} \lambda^2 \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right)^2 \right] \sqrt{-g}\, d^4x$$

The gravitational equation of motion $\delta S / \delta g_{\mu\nu} = 0$ is

$$- G_{\mu\nu} + \kappa T^{\text{gauge}}_{\mu\nu} + \kappa T^{\text{scalar}}_{\mu\nu} = 0 \tag{4.4}$$

where the energy-momentum tensors are

$$T^{\text{gauge}}_{\mu\nu} = \frac{\hbar}{g^2} \text{tr} \left(-2F_{\mu\sigma} F_\nu{}^\sigma + \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \tag{4.5}$$

$$T^{\text{scalar}}_{\mu\nu} = \hbar \left[2D_\mu \phi^\dagger D_\nu \phi - g_{\mu\nu} D_\sigma \phi^\dagger D^\sigma \phi - g_{\mu\nu} \frac{1}{2} \lambda^2 \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right)^2 \right] \tag{4.6}$$

4.2 Gauge and scalar equations of motion

Vary the action with respect to a variation of the gauge field $\delta B_\mu(x)$.

$$\delta S_{\text{gauge}} = \frac{\hbar}{2g^2} \int \text{tr} \left[4(D_\mu \delta B_\nu) F^{\mu\nu} \right] \sqrt{-g}\, d^4x$$

$$= \hbar \int \text{tr} \left[\delta B_\mu \left(\frac{2}{g^2} D_\nu F^{\mu\nu} \right) \right] \sqrt{-g}\, d^4x$$

$$\delta S_{\text{scalar}} = \hbar \int \left[-(\delta B_\mu \phi)^\dagger D^\mu \phi - D_\mu \phi^\dagger \delta B^\mu \phi \right] \sqrt{-g}\, d^4x \tag{4.7}$$

$$= \hbar \int \left[\phi^\dagger \delta B_\mu D^\mu \phi - D_\mu \phi^\dagger \delta B^\mu \phi \right] \sqrt{-g}\, d^4x$$

$$= \hbar \int \text{tr} \left[\delta B_\mu \left(D^\mu \phi \phi^\dagger - \phi D^\mu \phi^\dagger \right) \right] \sqrt{-g}\, d^4x$$

So the gauge field equation of motion is

$$\frac{1}{2} D^\nu F_{\mu\nu} + \frac{g^2}{4} \left(D_\mu \phi \phi^\dagger - \phi D_\mu \phi^\dagger \right) = 0 \tag{4.8}$$

Vary the scalar field by $\delta\phi$

$$\begin{aligned}\delta S_{\text{scalar}} &= \hbar \int \Big[-D_\mu \delta\phi^\dagger D^\mu \phi - D_\mu \phi^\dagger D^\mu \delta\phi \\ &\qquad -\lambda^2 \left(\phi^\dagger\phi - \frac{1}{2}v^2\right)\left(\delta\phi^\dagger\phi + \phi^\dagger\delta\phi\right)\Big]\sqrt{-g}\, d^4x \\ &= \hbar \int \delta\phi^\dagger \left[D_\mu D^\mu \phi - \lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)\phi\right]\sqrt{-g}\, d^4x \\ &\qquad + \delta\phi\hbar\left[D_\mu D^\mu \phi^\dagger - \lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)\phi^\dagger\right]\sqrt{-g}\, d^4x\end{aligned} \tag{4.9}$$

so the scalar field equation of motion is

$$-\frac{1}{2}D_\mu D^\mu \phi + \frac{1}{2}\lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)\phi = 0 \tag{4.10}$$

4.3 Spin(4)-symmetric energy-momentum tensors

The energy-momentum tensors of the Spin(4)-symmetric gauge and scalar fields are SO(4)-symmetric, so the energy-momentum tensors have the form

$$T_{\mu\nu}dx^\mu dx^\nu = \rho(T)a^2 dT^2 + p(T)a^2 ds^2_{S^3} \tag{4.11}$$

of perfect fluids of density ρ and pressure p.

For the Spin(4)-symmetric gauge field, calculate the energy-momentum tensor (4.5) at the north-pole. Use equation (3.24) for $F_{\mu\nu}$ at the north-pole.

$$\begin{aligned}F_{0a}(T,\hat{N}) &= -\frac{1}{2}\partial_T b\, i^{-1}\sigma_a \qquad F_{ab}(T,\hat{N}) = \frac{1}{2}(b^2-1)\epsilon_{abc}\, i^{-1}\sigma_c \\ \operatorname{tr}\left(F_{0\sigma}F_0{}^\sigma\right) &= a^{-2}\operatorname{tr}\left[\left(-\frac{1}{2}\partial_T b\, i^{-1}\sigma_a\right)\left(-\frac{1}{2}\partial_T b\, i^{-1}\sigma^a\right)\right] = -\frac{3}{2}a^{-2}(\partial_T b)^2 \\ \operatorname{tr}\left(F_{a\sigma}F_b{}^\sigma\right) &= a^{-2}\operatorname{tr}\left(-F_{a0}F_{b0} + F_{ac}F_b{}^c\right) \\ &= \frac{1}{4}a^{-2}(\partial_T b)^2\operatorname{tr}(\sigma_a\sigma_b) - \frac{1}{4}a^{-2}(b^2-1)^2\epsilon_{acd}\epsilon_{bce}\operatorname{tr}(\sigma_d\sigma_e) \\ &= \frac{1}{2}a^{-2}(\partial_T b)^2\delta_{ab} - a^{-2}(b^2-1)^2\delta_{ab} \\ \operatorname{tr}\left(F_{\rho\sigma}F^{\rho\sigma}\right) &= a^{-2}\operatorname{tr}\left(-F_{0\sigma}F_0{}^\sigma + F_{a\sigma}F^{b\sigma}\right) = 3a^{-4}\left[(\partial_T b)^2 - (b^2-1)^2\right]\end{aligned} \tag{4.12}$$

So at the north-pole

$$\begin{aligned}T_{00}^{\text{gauge}} &= \frac{\hbar}{g^2}\operatorname{tr}\left(-2F_{0\sigma}F_0{}^\sigma + \frac{1}{2}g_{00}F_{\rho\sigma}F^{\rho\sigma}\right) = \frac{3\hbar}{g^2a^2}\left[\frac{1}{2}(\partial_T b)^2 + \frac{1}{2}(b^2-1)^2\right] \\ &= \frac{3\hbar E_b}{g^2a^2}\end{aligned} \tag{4.13}$$

where

$$E_b = \frac{1}{2}(\partial_T b)^2 + \frac{1}{2}(b^2-1)^2 \tag{4.14}$$

The gauge energy-momentum tensor is traceless, $g^{\mu\nu}T^{\text{gauge}}_{\mu\nu} = 0$, so

$$\begin{aligned} T^{\text{gauge}}_{\mu\nu} dx^\mu dx^\nu &= \frac{3\hbar E_b}{g^2 a^2}\left(dT^2 + \frac{1}{3} ds^2_{S^3}\right) \\ \rho_{\text{gauge}} = \frac{3\hbar E_b}{g^2 a^4} &\qquad p_{\text{gauge}} = \frac{1}{3}\rho_{\text{gauge}} \end{aligned} \tag{4.15}$$

For the Spin(4)-symmetric scalar field $\phi = 0$, the energy-momentum tensor (4.6) is

$$\begin{aligned} T^{\text{scalar}}_{\mu\nu} &= \frac{\hbar\lambda^2 v^4}{8}(-g_{\mu\nu}) \\ T^{\text{scalar}}_{\mu\nu} dx^\mu dx^\nu &= \mathcal{E}_0 \left(a^2 dT^2 - a^2 ds^2_{S^3}\right) \\ \rho_{\text{scalar}} = \mathcal{E}_0 \qquad p_{\text{scalar}} &= -\rho_{\text{scalar}} \qquad \mathcal{E}_0 = \frac{\hbar\lambda^2 v^4}{8} \end{aligned} \tag{4.16}$$

4.4 Spin(4)-symmetric gravitational equation of motion

The gravitational equation of motion (4.4)

$$G_{\mu\nu} = \kappa T^{\text{gauge}}_{\mu\nu} + \kappa T^{\text{scalar}}_{\mu\nu} \tag{4.17}$$

becomes two scalar equations on the Spin(4)-symmetric fields, one equation between the coefficients of dT^2, the other equation between the coeffients of $ds^2_{S^3}$. Combining formula (3.4) for the Einstein tensor,

$$G_{\mu\nu} dx^\mu dx^\nu = \left[3a^{-2}(\partial_T a)^2 + 3\right] dT^2 + \left[a^{-2}(\partial_T a)^2 - 2a^{-1}\partial_T^2 a - 1\right] ds^2_{S^3} \tag{4.18}$$

with formulas (4.15) and (4.16) for the energy-momentum tensors

$$\begin{aligned} T^{\text{gauge}}_{\mu\nu} dx^\mu dx^\nu &= \rho_{\text{gauge}} a^2 \left(dT^2 + \frac{1}{3} ds^2_{S^3}\right) \\ T^{\text{scalar}}_{\mu\nu} dx^\mu dx^\nu &= \rho_{\text{scalar}} a^2 \left(dT^2 - ds^2_{S^3}\right) \\ \rho_{\text{gauge}} = \frac{3\hbar E_b}{g^2 a^4} &\qquad E_b = \frac{1}{2}(\partial_T b)^2 + \frac{1}{2}(b^2 - 1)^2 \\ \rho_{\text{scalar}} = \mathcal{E}_0 &\qquad \mathcal{E}_0 = \frac{\hbar\lambda^2 v^4}{8} \end{aligned} \tag{4.19}$$

gives the gravitational equations of motion

$$\begin{aligned} 3a^{-2}(\partial_T a)^2 + 3 &= \kappa\rho_{\text{gauge}} a^2 + \kappa\rho_{\text{scalar}} a^2 \\ a^{-2}(\partial_T a)^2 - 2a^{-1}\partial_T^2 a - 1 &= \frac{1}{3}\kappa\rho_{\text{gauge}} a^2 - \kappa\rho_{\text{scalar}} a^2 \end{aligned} \tag{4.20}$$

Re-write the first equation as

$$\begin{aligned} \frac{1}{2}(\partial_T a)^2 + \frac{1}{2}a^2 - \frac{1}{6}\kappa\rho_{\text{scalar}} a^4 &= \frac{1}{6}\kappa\rho_{\text{gauge}} a^4 \\ \frac{1}{2}(\partial_T a)^2 + \frac{1}{2}a^2 - \frac{\kappa\mathcal{E}_0}{6} a^4 &= \frac{\kappa\hbar}{2g^2} E_b \end{aligned} \tag{4.21}$$

Re-write 1/3 times the first equation minus the second equation

$$\begin{aligned} 2a^{-1}\partial_T^2 a + 2 &= \frac{4}{3}\kappa\rho_{\text{scalar}}a^2 \\ \partial_T^2 a + a - \frac{2}{3}\kappa\mathcal{E}_0 a^3 &= 0 \end{aligned} \tag{4.22}$$

The gravitational equations of motion become

$$\begin{aligned} \frac{1}{2}(\partial_T a)^2 + \frac{1}{2}a^2 - \frac{\kappa\mathcal{E}_0}{6}a^4 &= \frac{\kappa\hbar}{2g^2}E_b \\ \partial_T^2 a + a - \frac{2}{3}\kappa\mathcal{E}_0 a^3 &= 0 \end{aligned} \tag{4.23}$$

In terms of the dimensionless scale factor $\hat{a} = a/t_{\text{I}}$,

$$\begin{aligned} \frac{1}{2}(\partial_T \hat{a})^2 + \frac{1}{2}\hat{a}^2 - \frac{\kappa\mathcal{E}_0 t_{\text{I}}^2}{6}\hat{a}^4 &= \frac{\kappa\hbar}{2g^2 t_{\text{I}}^2}E_b \\ \partial_T^2 \hat{a} + \hat{a} - \frac{2}{3}\kappa\mathcal{E}_0 t_{\text{I}}^2 \hat{a}^3 &= 0 \end{aligned} \tag{4.24}$$

Use the definitions

$$\frac{\kappa\mathcal{E}_0 t_{\text{I}}^2}{3} = 1 \qquad \epsilon_{\text{w}}^{-4} = \frac{2g^2 t_{\text{I}}^2}{\kappa\hbar} \tag{4.25}$$

the gravitational equations of motion are

$$\begin{aligned} E_{\hat{a}} &= \epsilon_{\text{w}}^4 E_b \\ \partial_T^2 \hat{a} + \hat{a} - 2\hat{a}^3 &= 0 \end{aligned} \tag{4.26}$$

where

$$E_{\hat{a}} = \frac{1}{2}(\partial_T \hat{a})^2 + \frac{1}{2}\hat{a}^2 - \frac{1}{2}\hat{a}^4 \tag{4.27}$$

4.5 Spin(4)-symmetric gauge and scalar equations of motion

For the Spin(4)-symmetric gauge field $B = (1 + b)\gamma$ and scalar field $\phi = 0$, the scalar equation of motion (4.10) is trivially satisfied and the gauge equation of motion (4.8) is

$$D^\nu F_{\mu\nu} = 0 \tag{4.28}$$

The latter is

$$\begin{aligned} 0 = \delta\frac{1}{\hbar}S_{\text{gauge}} &= \delta\frac{6\pi^2}{g^2}\int \frac{1}{2}\left[(\partial_T b)^2 - (b^2-1)^2\right]\, dT \\ &= \delta\frac{6\pi^2}{g^2}\int \left[-\partial_T^2 b - (b^2-1)2b\right]\delta b\, dT \end{aligned} \tag{4.29}$$

The gauge equation of motion (4.28) is the equation of motion

$$-\partial_T^2 b - (b^2-1)2b = 0 \tag{4.30}$$

derived from the Spin(4)-symmetric action.

Alternatively, use the Hodge-$*$ operator

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\nu\sigma}F_{\nu\sigma} \qquad *^2 = -1 \tag{4.31}$$

to re-write the equation of motion (4.28) as

$$D{*}F = d{*}F + [B,\, {*}F] = 0 \tag{4.32}$$

The curvature 2-form of the Spin(4)-symmetric gauge field $B = (1+b)\gamma$ is (3.23)

$$F = (\partial_T b) dT \wedge \gamma + (b^2 - 1)\gamma \wedge \gamma \tag{4.33}$$

At the north-pole,

$$\begin{aligned} {*}(dT \wedge d\hat{x}^a) &= \frac{1}{2}\epsilon_{abc} d\hat{x}^b \wedge d\hat{x}^c \\ {*}dT \wedge \gamma(\hat{N}) = {*}dT \wedge \left(\frac{1}{2} d\hat{x}^a i\sigma_a\right) &= \frac{1}{4}\epsilon_{abc} d\hat{x}^b \wedge d\hat{x}^c i\sigma_a = -\gamma \wedge \gamma(\hat{N}) \end{aligned} \tag{4.34}$$

so

$$* \,(dT \wedge \gamma) = -\gamma \wedge \gamma \qquad * \,(\gamma \wedge \gamma) = dT \wedge \gamma \tag{4.35}$$

so, using the identity $d\gamma = -2\gamma \wedge \gamma$,

$$\begin{aligned} {*}F &= (b^2 - 1) dT \wedge \gamma - (\partial_T b)\gamma \wedge \gamma \\ d{*}F &= 2(b^2 - 1) dT \wedge \gamma \wedge \gamma - (\partial_T^2 b) dT \wedge \gamma \wedge \gamma \\ [B,\, {*}F] &= (1+b)(b^2-1)(\gamma \wedge dT \wedge \gamma - dT \wedge \gamma \wedge \gamma) \\ &= -2(1+b)(b^2-1) dT \wedge \gamma \wedge \gamma \\ D{*}F &= \left(-\partial_T^2 b - 2b(b^2-1)\right) dT \wedge \gamma \wedge \gamma \end{aligned} \tag{4.36}$$

So the equation of motion $D{*}F = 0$ is the b oscillator equation of motion (4.30).

5 Numbers

5.1 Fundamental constants

From *2018 CODATA Recommended values of the fundamental constants of physics and chemistry,* NIST SP 959 (June 2019) [1]

Defining constants of the International System of Units (SI)

$$\begin{aligned} c &= 2.99792458 \times 10^{8}\,\mathrm{ms}^{-1} \\ \hbar &= 1.054571817 \times 10^{-34}\,\mathrm{Js} \\ e &= 1.602176634 \times 10^{-19}\,\mathrm{C} \\ k_{\mathrm{B}} &= 1.380649 \times 10^{-23}\,\mathrm{J\,K}^{-1} \end{aligned} \tag{5.1}$$

so

$$1\,\mathrm{GeV} = 1.602176634 \times 10^{-10}\,\mathrm{J} = k_{\mathrm{B}}\,(1.160452 \times 10^{13}\,\mathrm{K}) \tag{5.2}$$

Newtonian constant of gravitation

$$\begin{aligned} G &= 6.67430(15) \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2} \\ \kappa = 8\pi G &= 1.10982 \times 10^{-61}\,\mathrm{s\,GeV}^{-1}\,c^5 \end{aligned} \tag{5.3}$$

5.2 Standard Model coupling constants

From Particle Data Group, *Review of Particle Physics 2018 and 2019 update* [2]

- Section 1. Physical Constants (p. 127)
- Section 10. Electroweak Model and Constraints on New Physics (p. 161)

G_F and m_W are from Section 1 (p. 127). m_H is from the 2019 update.

$$\begin{aligned} G_F &= 1.1663787(6)\times 10^{-5}\,\mathrm{GeV}^{-2} \\ m_W &= 80.379(12)\,\mathrm{GeV} \\ m_H &= 125.10(14)\,\mathrm{GeV} \end{aligned} \tag{5.4}$$

In Section 10, the Higgs potential is written (in $\hbar = 1$ units)

$$V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda^2}{2}(\phi^\dagger\phi)^2 \tag{5.5}$$

(actually μ^2 is written there without the sign). This is the same as the Higgs potential in (1.1) up to the constant term. The couplings are related to measurements by

$$\langle\phi\rangle_{\mathrm{vac}} = \frac{v}{\sqrt{2}} \qquad m_H = \lambda\hbar v \qquad m_W = \frac{1}{2}g\hbar v \qquad \frac{G_F}{\sqrt{2}} = \frac{1}{2(\hbar v)^2} \tag{5.6}$$

which gives

$$\begin{aligned} \hbar v &= 2^{-\frac{1}{4}}G_F^{-\frac{1}{2}} &&= 246\,\mathrm{GeV} \\ g &= \frac{2m_W}{\hbar v} &&= 0.653 \\ \lambda &= \frac{m_H}{\hbar v} &&= 0.508 \end{aligned} \tag{5.7}$$

5.3 Gravitational and weak time scales t_{grav}, t_W

Define the gravitational time scale

$$t_{\mathrm{grav}} = (\hbar\kappa)^{\frac{1}{2}} = (8\pi\hbar G)^{\frac{1}{2}} = (8\pi)^{\frac{1}{2}}t_{\mathrm{P}} = 5.01\,t_{\mathrm{P}} = 2.70\times 10^{-43}\,\mathrm{s} \tag{5.8}$$

where $t_{\mathrm{P}} = (\hbar G)^{\frac{1}{2}}$ is the Planck time.

Define the weak time scale

$$t_W = \frac{2}{gv} = \frac{\hbar}{m_W} = \frac{\hbar}{80.4\,\mathrm{GeV}} = 8.19\times 10^{-27}\,\mathrm{s} \tag{5.9}$$

5.4 The scalar field energy density $\mathcal{E}_0$

$$\frac{1}{\hbar}\mathcal{E}_0 = \frac{1}{8}\lambda^2 v^4 = \frac{1}{8}\lambda^2\left(\frac{2}{g}\frac{1}{t_W}\right)^4 = \frac{2\lambda^2}{g^4}t_W^{-4} = 2.84\,t_W^{-4} \tag{5.10}$$

5.5 Seesaw time scale t_{I}

$$\begin{aligned} t_{\mathrm{I}}^2 &= \frac{3}{\kappa\mathcal{E}_0} = \frac{24}{\kappa\hbar\lambda^2 v^4} = \frac{1}{\kappa\hbar}\frac{3g^4}{2\lambda^2}t_W^4 = \frac{3g^4}{2\lambda^2}\frac{t_W^4}{t_{\mathrm{grav}}^2} \\ t_{\mathrm{I}} &= \sqrt{\frac{3}{2}}\frac{g^2}{\lambda}\frac{t_W^2}{t_{\mathrm{grav}}} = 1.03\,\frac{t_W^2}{t_{\mathrm{grav}}} = 2.55\times 10^{-10}\,\mathrm{s} = 7.64\,\mathrm{cm} \end{aligned} \tag{5.11}$$

5.6 Seesaw ratio ϵ_{w}

$$\epsilon_{\mathrm{w}}^4 = \frac{\kappa\hbar}{2g^2 t_{\mathrm{I}}^2} = \frac{\kappa\hbar}{2g^2}\frac{\kappa\hbar\lambda^2 v^4}{24} = \frac{\kappa^2\hbar^2\lambda^2 v^4}{48g^2} \qquad \epsilon_{\mathrm{w}}^2 = \frac{\kappa\hbar\lambda v^2}{4\cdot 3^{1/2} g} \tag{5.12}$$

$$\epsilon_{\mathrm{w}} = 3.39\times 10^{-17} \tag{5.13}$$

$$\begin{aligned}
\epsilon_{\mathrm{w}}^4 &= \frac{\kappa\hbar}{2g^2 t_{\mathrm{I}}^2} = \frac{1}{2g^2}\frac{t_{\mathrm{grav}}^2}{t_{\mathrm{I}}^2} & \epsilon_{\mathrm{w}} &= \left(\frac{1}{2g^2}\right)^{\frac{1}{4}}\left(\frac{t_{\mathrm{grav}}}{t_{\mathrm{I}}}\right)^{\frac{1}{2}} = 1.04\left(\frac{t_{\mathrm{grav}}}{t_{\mathrm{I}}}\right)^{\frac{1}{2}} \\
&= \frac{t_{\mathrm{grav}}^2}{2g^2}\left(\frac{2\lambda^2}{3g^4}\frac{t_{\mathrm{grav}}^2}{t_W^4}\right) & \epsilon_{\mathrm{w}} &= \left(\frac{\lambda^2}{3g^6}\right)^{\frac{1}{4}}\frac{t_{\mathrm{grav}}}{t_W} \qquad = 1.03\,\frac{t_{\mathrm{grav}}}{t_W} \\
&= \left(\frac{1}{2g^2}\frac{t_{\mathrm{grav}}^2}{t_{\mathrm{I}}^2}\right)^2\left(\frac{\lambda^2}{3g^6}\frac{t_{\mathrm{grav}}^4}{t_W^4}\right)^{-1} & \epsilon_{\mathrm{w}} &= \left(\frac{3g^2}{4\lambda^2}\right)^{\frac{1}{4}}\frac{t_W}{t_{\mathrm{I}}} \qquad = 1.05\,\frac{t_W}{t_{\mathrm{I}}}
\end{aligned} \tag{5.14}$$

5.7 Units of action for the two oscillators

In the action (3.35) for the $\hat{a}$ and b oscillators,

$$S = \frac{6\pi^2}{g^2}\hbar\int\left(-\epsilon_{\mathrm{w}}^{-4}\left[\frac{1}{2}(\partial_T\hat{a})^2 - V_{\hat{a}}(\hat{a})\right] + \left[\frac{1}{2}(\partial_T b)^2 - V_b(b)\right]\right) dT \tag{5.15}$$

The units of action for the b oscillator are

$$\hbar_b = \frac{6\pi^2}{g^2}\hbar = 139\,\hbar \tag{5.16}$$

For the $\hat{a}$ oscillator the units of action are

$$\hbar_{\hat{a}} = \frac{6\pi^2}{g^2}\epsilon_{\mathrm{w}}^{-4}\hbar = 139\epsilon_{\mathrm{w}}^{-4}\,\hbar \tag{5.17}$$

6 Stability of $\phi = 0$

Expand the scalar field action (1.1)

$$\frac{1}{\hbar}S_{\mathrm{scalar}} = \int\left[-D_\mu\phi^\dagger D^\mu\phi - \frac{1}{2}\lambda^2\left(\phi^\dagger\phi - \frac{1}{2}v^2\right)^2\right]\sqrt{-g}\,d^4x \tag{6.1}$$

in powers of a perturbation $\phi(x)$ around $\phi = 0$,

$$\frac{1}{\hbar}S_{\mathrm{scalar}}(\phi) = \frac{1}{\hbar}S_{\mathrm{scalar}}(0) + \frac{1}{\hbar}S_{\mathrm{scalar}}^{(2)}(\phi) + O(\phi^4) \tag{6.2}$$

$$\begin{aligned}
\frac{1}{\hbar}S_{\mathrm{scalar}}^{(2)} &= \int\left[-\left(D_\mu\phi\right)^\dagger\left(D^\mu\phi\right) + \frac{1}{2}\lambda^2 v^2\phi^\dagger\phi\right]\sqrt{-g}\,d^4x \\
&= \int\left[a^{-2}\left(\partial_T\phi\right)^\dagger\left(\partial_T\phi\right) - a^{-2}\mathcal{V}(\phi)\right]a^4 d^3\hat{x}dT
\end{aligned} \tag{6.3}$$

$$\mathcal{V}(\phi) = \hat{g}^{jk}\left(D_j\phi\right)^\dagger\left(D_k\phi\right) - a^2\frac{1}{2}\lambda^2 v^2\phi^\dagger\phi \tag{6.4}$$

where $\hat{g}_{jk}$ is the metric of the unit 3-sphere, $ds^2_{S^3} = \hat{g}_{jk}(\hat{x})d\hat{x}^j d\hat{x}^k$ and $D_0 = \partial_T$, $D_k = \partial_k + B_k$ is the gauge covariant derivative, with $B_k = (1+b)\gamma_k$.

Stability of the $\phi = 0$ solution is the condition that

$$0 \leq \int_{S^3} \left[\hat{g}^{jk} \left(D_j\phi\right)^\dagger \left(D_k\phi\right) - a^2 \frac{1}{2}\lambda^2 v^2 \phi^\dagger \phi \right] d^3\hat{x} \tag{6.5}$$

for all perturbations $\phi(x)$. Integrate by parts to write this

$$0 \leq \int_{S^3} \phi^\dagger \left[\hat{g}^{jk} D_j^\dagger D_k - \frac{1}{2} a^2 \lambda^2 v^2 \right] \phi \, d^3\hat{x} \tag{6.6}$$

so the stability condition is the operator condition

$$0 \leq \hat{g}^{jk} D_j^\dagger D_k - \frac{1}{2} a^2 \lambda^2 v^2 \tag{6.7}$$

Stability at time scales much longer than the b oscillation period is the time average condition

$$0 \leq \langle \hat{g}^{jk} D_j^\dagger D_k \rangle - \frac{1}{2} a^2 \lambda^2 v^2 \tag{6.8}$$

6.1 Dirac matrices and spinors on S^3

The Spin(4)-symmetric $\mathfrak{su}(2)$-valued 1-form γ defined by (2.3) and (2.30)

$$\begin{aligned} g_{\hat{x}} &= \hat{x}^4 \mathbf{1} + \hat{x}^a i^{-1} \sigma_a \\ \gamma(\hat{x}) &= -\frac{1}{2} \left(d g_{\hat{x}}\right) g_{\hat{x}}^{-1} = \frac{1}{2} g_{\hat{x}} d g_{\hat{x}}^\dagger \end{aligned} \tag{6.9}$$

At the north-pole $\hat{N}$,

$$\gamma_a = \frac{1}{2} i \sigma_a \qquad \gamma_a \gamma_b = -\frac{1}{4}\delta_{ab} - \frac{1}{4} i \epsilon_{abc}\sigma_c = -\frac{1}{4}\hat{g}_{ab} - \frac{1}{2}\epsilon_{ab}{}^c \gamma_c \tag{6.10}$$

so everywhere on S^3

$$\gamma_j^\dagger = -\gamma_j \qquad \gamma_j \gamma_k = -\frac{1}{4}\hat{g}_{jk} - \frac{1}{2}\epsilon_{jk}{}^i \gamma_i \qquad \hat{g}^{jk}\gamma_j^\dagger \gamma_k = \hat{g}^{jk}(-\gamma_j)\gamma_k = \frac{3}{4}\mathbf{1} \tag{6.11}$$

so the matrices $2i\gamma_j(\hat{x})$ can be interpreted as the Dirac matrices on S^3. The doublet scalar field ϕ can be interpreted geometrically as a spinor on S^3.

6.2 Time-averaged stability

Write

$$D_k = D_k^0 + b\gamma_k \qquad D_k^0 = \partial_k + \gamma_k \tag{6.12}$$

The b oscillations are symmetric in $b \to -b$ so the time average $\langle b \rangle = 0$ so

$$\begin{aligned} \langle \hat{g}^{jk} D_j^\dagger D_k \rangle &= \langle \hat{g}^{jk} (D_j^{0\dagger} + b\gamma_j^\dagger)(D_k^0 + b\gamma_k) \rangle \\ &= \hat{g}^{jk} D_j^{0\dagger} D_k^0 + \langle b^2 \rangle \hat{g}^{jk} \gamma_j^\dagger \gamma_k \\ &= \hat{g}^{jk} D_j^{0\dagger} D_k^0 + \frac{3}{4}\langle b^2 \rangle \end{aligned} \tag{6.13}$$

So the time-average stability condition is

$$0 \leq \hat{g}^{jk} D_j^{0\dagger} D_k^0 + \frac{3}{4}\langle b^2 \rangle - \frac{1}{2} a^2 \lambda^2 v^2 \tag{6.14}$$

6.3 Spinor covariant derivative

Let ∇_j be the covariant derivative on vectors and tensors on the unit 3-sphere. Then

$$\nabla_j \gamma_k = \frac{1}{2}\partial_j g_{\hat{x}} \partial_k g_{\hat{x}}^\dagger + \frac{1}{2} g_{\hat{x}} \nabla_j \partial_k g_{\hat{x}}^\dagger \tag{6.15}$$

Calculate at the north-pole $\hat{N} = (0,0,0,1)$,

$$\partial_a \partial_b g_{\hat{x}}^\dagger(\hat{N}) = \partial_a \partial_b \left(\hat{x}^4 \mathbf{1} + \hat{x}^a i\sigma_a\right)_{/\hat{x}^a=0} = -\delta_{ab}\mathbf{1} \tag{6.16}$$

so everywhere on S^3

$$\nabla_j \partial_k g_{\hat{x}}^\dagger = -\hat{g}_{jk} \tag{6.17}$$

Use this in (6.15) along with

$$\partial_j g_{\hat{x}} \partial_k g_{\hat{x}}^\dagger = (-2\gamma_j g_{\hat{x}})(2 g_{\hat{x}}^{-1}\gamma_k) = -4\gamma_j\gamma_k = \hat{g}_{jk} + 2\epsilon_{jk}{}^i \gamma_i \tag{6.18}$$

to get

$$\nabla_j \gamma_k = \epsilon_{jk}{}^i \gamma_i \tag{6.19}$$

Now extend the gauge covariant derivative $D_j^0 = \partial_j + \nabla_j$ to tensors with values in the fundamental bundle as

$$D_j^0 = \nabla_j + \gamma_j \tag{6.20}$$

Then

$$D_j^0 \gamma_k = \nabla_j \gamma_k + [\gamma_j,\, \gamma_k] = \epsilon_{jk}{}^i\gamma_i - \epsilon_{jk}{}^i \gamma_i = 0 \tag{6.21}$$

which is to say that the Dirac matrices are covariant constant under D_j^0. Therefore D_j^0 is the spinor covariant derivative.

The curvature form of the spinor covariant derivative is

$$\begin{aligned}[D_j^0,\, D_k^0] &= [\nabla_j + \gamma_j,\, \nabla_k + \gamma_k] = [\nabla_j,\, \nabla_k] + \nabla_j\gamma_k - \nabla_k \gamma_j + [\gamma_j,\, \gamma_k] \\ &= [\nabla_j,\, \nabla_k] + \epsilon_{jk}{}^i \gamma_i\end{aligned} \tag{6.22}$$

The "spinors" ϕ are functions with values in $\mathbb{C}^2$, so $[\nabla_j,\, \nabla_k]\phi = 0$, so

$$[D_j^0,\, D_k^0]\phi = \epsilon_{jk}{}^i \gamma_i \phi \tag{6.23}$$

6.4 Dirac operator

The Dirac operator on S^3 is

$$\not{D} = 2\hat{g}^{jk}\gamma_j D_k^0 = 2\gamma^k D_k^0 \qquad \not{D}^\dagger = \not{D} \tag{6.24}$$

Calculate $\not{D}^2$ acting on "spinors" ϕ

$$\begin{aligned}\not{D}^2 &= 4\gamma^j D_j^0 \gamma^k D_k^0 = 4\gamma^j\gamma^k D_j^0 D_k^0 = (-\hat{g}^{jk} - 2\epsilon^{jki}\gamma_i) D_j^0 D_k^0 \\ &= \hat{g}^{jk} D_j^{0\dagger} D_k^0 - \epsilon^{jki}\gamma_i [D_j^0\, D_k^0] = \hat{g}^{jk} D_j^{0\dagger} D_k^0 - \epsilon^{jki}\gamma_i \epsilon_{jk}{}^{i'}\gamma_{i'} \\ &= \hat{g}^{jk} D_j^{0\dagger} D_k^0 + 2\hat{g}^{ii'}\gamma_i\gamma_{i'} = \hat{g}^{jk} D_j^{0\dagger} D_k^0 + 2\hat{g}^{ii'}\hat{g}_{ii'} \\ &= \hat{g}^{jk} D_j^{0\dagger} D_k^0 + \frac{3}{2}\end{aligned} \tag{6.25}$$

then calculate the gauge laplacian acting on "spinors" ϕ

$$\begin{aligned}\hat{g}^{jk}D_j^\dagger D_k &= -\hat{g}^{jk}(D_j^0+b\gamma_j)(D_k^0+b\gamma_k) = -\hat{g}^{jk}D_j^0D_k^0 - 2b\gamma^k D_k^0 - b^2\hat{g}^{jk}\gamma_j\gamma_k \\ &= \not{D}^2 - \frac{3}{2} - b\not{D} + \frac{3}{4}b^2 \\ &= \left(\not{D}-\frac{1}{2}b\right)^2 - \frac{3}{2} + \frac{1}{2}b^2\end{aligned} \tag{6.26}$$

so there is a bound

$$0 \le \left(\not{D}-\frac{1}{2}b\right)^2 - \frac{3}{2} + \frac{1}{2}b^2 \tag{6.27}$$

Minimize the rhs wrt b

$$2\left(\not{D}-\frac{1}{2}b\right)\left(-\frac{1}{2}\right) + b = 0 \qquad b = \frac{2}{3}\not{D} \tag{6.28}$$

to get the bound

$$0 \le \left(\not{D}-\frac{1}{2}\frac{2}{3}\not{D}\right)^2 - \frac{3}{2} + \frac{1}{2}\left(\frac{2}{3}\not{D}\right)^2 = \frac{2}{3}\not{D}^2 - \frac{3}{2} \tag{6.29}$$

or

$$\left(\frac{3}{2}\right)^2 \le \not{D}^2 \tag{6.30}$$

Equality, $\not{D} = \pm 3/2$, requires $b = \pm 1$. Equality is equivalent to $\hat{g}^{jk}D_j^\dagger D_k = 0$ which is equivalent to $D_k = 0$. For $b = -1$,

$$D_k\phi = (\partial_k + \gamma_k + b\gamma_k)\phi = \partial_k\phi \tag{6.31}$$

so the zero mode of D_k is $\phi = \phi_0$ a constant. When $b = 1$,

$$D_k\phi = (\partial_k + 2\gamma_k)\phi = (\partial_k + g_{\hat{x}}dg_{\hat{x}}^\dagger)\phi = g_{\hat{x}}\partial_k(g_{\hat{x}}^{-1}\phi) \tag{6.32}$$

so the zero mode of D_k is then $\phi = g_{\hat{x}}\phi_0$ with ϕ_0 constant.

6.5 Parity operator

Let the parity operator $P \in O(4)$ be

$$P\colon (\hat{x}^i, \hat{x}^4) \mapsto (-\hat{x}^i, \hat{x}^4) \tag{6.33}$$

so

$$g_{P\hat{x}} = g_{\hat{x}}^{-1} \tag{6.34}$$

Let P act on the fundamental SU(2) bundle by

$$P\phi(\hat{x}) = g_{\hat{x}}\phi(P\hat{x}) \tag{6.35}$$

P has the following properties

$$\begin{aligned}P^2\phi(\hat{x}) &= P\left(g_{\hat{x}}\phi(P\hat{x})\right) = g_{\hat{x}}g_{P\hat{x}}\phi(P^2\hat{x}) = \phi(\hat{x}) \\ P\gamma P^{-1} &= P\frac{1}{2}g_{\hat{x}}d\left(g_{\hat{x}}^{-1}\right)P^{-1} = \frac{1}{2}g_{\hat{x}}g_{P\hat{x}}d\left(g_{P\hat{x}}^{-1}\right)g_{\hat{x}}^{-1} = \frac{1}{2}(dg_{\hat{x}})g_{\hat{x}}^{-1} = -\gamma \\ PD^0P^{-1} &= P(d+\gamma)P^{-1} = g_{\hat{x}}dg_{\hat{x}}^{-1} - \gamma = d + g_{\hat{x}}d\left(g_{\hat{x}}^{-1}\right) - \gamma = d + \gamma = D^0\end{aligned} \tag{6.36}$$

which are

$$P^2 = 1 \qquad P\gamma P^{-1} = -\gamma \qquad PD^0P^{-1} = D^0 \tag{6.37}$$

These imply

$$P\not{D}P^{-1} = -\not{D} \tag{6.38}$$

and

$$PDP^{-1} = P(D^0 + b\gamma)P^{-1} = D^0 - b\gamma \tag{6.39}$$

so the SU(2) gauge field $b(T)$ is parity-odd.

6.6 Time-averaged stability (II)

Combine the identity (6.25) with the bound (6.30)

$$\left(\frac{3}{2}\right)^2 \leq \not{D}^2 = \hat{g}^{jk} D_j^{0\dagger} D_k^0 + \frac{3}{2} \tag{6.40}$$

to get the bound

$$\frac{3}{4} \leq \hat{g}^{jk} D_j^{0\dagger} D_k^0 \tag{6.41}$$

with the lower bound realized on the constants $\phi = \phi_0$ and on their gauge transforms $\phi = P\phi_0 = g_{\hat{x}}\phi_0$. So the time-averaged stability condition (6.14) is

$$0 \leq \frac{3}{4} + \frac{3}{4}\langle b^2 \rangle - \frac{1}{2} a^2 \lambda^2 v^2 \tag{6.42}$$

The stability condition is first violated when a reaches a_{EW} given by equality in (6.42)

$$\frac{2}{3}\lambda^2 v^2 a_{\mathrm{EW}}^2 = 1 + \langle b^2 \rangle \tag{6.43}$$

When a is slightly greater than a_{EW}, the zero modes $\phi = \phi_0$ and their gauge transforms $\phi = P\phi_0$ are the only unstable modes. The other modes of $\phi(x)$ become unstable at larger values of a.

7 Solution of the b oscillator by an elliptic function

7.1 Reparametrize T and b

The energy equation for the b oscillator is

$$\frac{1}{2}\left(\frac{db}{dT}\right)^2 + \frac{1}{2}(b^2 - 1)^2 = E_b \tag{7.1}$$

b oscillates between $\pm b_{\mathrm{max}}$ where

$$\frac{1}{2}(b_{\mathrm{max}}^2 - 1)^2 = E_b \qquad b_{\mathrm{max}}^2 - 1 = (2E_b)^{\frac{1}{2}} \tag{7.2}$$

Write the energy equation

$$\left(\frac{db}{dT}\right)^2 = 2E_b - (b^2 - 1)^2 = (b_{\mathrm{max}}^2 - 1)^2 - (b^2 - 1)^2 = (b_{\mathrm{max}}^2 - b^2)(b_{\mathrm{max}}^2 - 2 + b^2) \tag{7.3}$$

Change variables from T and $b(T)$ to z and $y(z)$

$$T = \epsilon_b z \qquad b = b_{\max} y \tag{7.4}$$

with ϵ_b to be determined. Now the energy equation is

$$\begin{aligned} \frac{b_{\max}^2}{\epsilon_b^2}\left(\frac{dy}{dz}\right)^2 &= (b_{\max}^2 - b_{\max}^2 y^2)(b_{\max}^2 - 2 + b_{\max}^2 y^2) \\ \left(\frac{dy}{dz}\right)^2 &= (1 - y^2)(\epsilon_b^2 b_{\max}^2 - 2\epsilon_b^2 + \epsilon_b^2 b_{\max}^2 y^2) \end{aligned} \tag{7.5}$$

which is

$$\left(\frac{dy}{dz}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2) \tag{7.6}$$

for

$$1 - k^2 = \epsilon_b^2 b_{\max}^2 - 2\epsilon_b^2 \qquad k^2 = \epsilon_b^2 b_{\max}^2 \tag{7.7}$$

The sum of these last two equations gives

$$1 = 2\epsilon_b^2 b_{\max}^2 - 2\epsilon_b^2 = 2\epsilon_b^2(b_{\max}^2 - 1) = 2\epsilon_b^2(2E_b)^{\frac{1}{2}} \qquad \epsilon_b = (8E_b)^{-\frac{1}{4}} \tag{7.8}$$

The difference of the two equations gives

$$2k^2 - 1 = 2\epsilon_b^2 \qquad k^2 = \frac{1}{2} + \epsilon_b^{-2} \tag{7.9}$$

7.2 The Jacobi elliptic function $\mathrm{cn}(z, k)$

The energy equation (7.6)

$$\left(\frac{dy}{dz}\right)^2 = (1 - y^2)(1 - k^2 + k^2 y^2) \tag{7.10}$$

is solved by the Jacobi elliptic function [3, 4]

$$y = \mathrm{cn}(z, k) \tag{7.11}$$

by integrating

$$\begin{aligned} &\frac{dy}{dz} = -\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)} \qquad dz = \frac{-dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)}} \\ &z = \int_{\mathrm{cn}(z,k)}^{1} \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)}} \end{aligned} \tag{7.12}$$

7.3 Properties of the Jacobi elliptic functions

The Jacobi elliptic functions [3, 4] are $\mathrm{sn}(z, k)$, $\mathrm{cn}(z, k)$, $\mathrm{dn}(z, k)$.

Differential equations:

y	$\left(\frac{dy}{dz}\right)^2 =$	$\frac{d^2y}{dz^2} =$
$\mathrm{sn}(z, k)$	$(1 - y^2)(1 - k^2 y^2)$	$-(1 + k^2)y + 2k^2 y^3$
$\mathrm{cn}(z, k)$	$(1 - y^2)(1 - k^2 + k^2 y^2)$	$-(1 - 2k^2)y - 2k^2 y^3$
$\mathrm{dn}(z, k)$	$(y^2 - 1)(1 - k^2 - y^2)$	$(2 - k^2)y - 2y^3$

(7.13)

Derivatives:

$$\mathrm{sn}' = \mathrm{cn}\,\mathrm{dn} \qquad \mathrm{cn}' = -\,\mathrm{sn}\,\mathrm{dn} \qquad \mathrm{dn}' = -k^2\,\mathrm{sn}\,\mathrm{cn} \tag{7.14}$$

Reflections:

$$\mathrm{sn}(-z) = -\,\mathrm{sn}(z) \qquad \mathrm{cn}(-z) = \mathrm{cn}(z) \qquad \mathrm{dn}(-z) = \mathrm{dn}(z) \tag{7.15}$$

Algebraic relations:

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1 \qquad k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1 \qquad k^2\,\mathrm{cn}^2 = \mathrm{dn}^2 + k^2 - 1 \tag{7.16}$$

A doubling identity:

$$\left(\frac{\mathrm{sn}\,\mathrm{dn}}{\mathrm{cn}}\right)^2(z,k) = \frac{1-\mathrm{cn}(2z,k)}{1+\mathrm{cn}(2z,k)} \tag{7.17}$$

In the following,

$$k^2 + k'^2 = 1 \qquad K = K(k) \qquad K' = K(k') \tag{7.18}$$

where $K(k)$ is the complete elliptic integral of the first kind.

Poles and zeros:

	pole	zero
sn	iK'	0
cn	iK'	K
dn	iK'	$K + iK'$

$$+\,2mK + 2inK' \tag{7.19}$$

Half-periods:

$$\begin{aligned} \mathrm{sn}(z + 2Km + 2K'm'i) &= (-1)^m\,\mathrm{sn}(z) \\ \mathrm{cn}(z + 2Km + 2K'm'i) &= (-1)^{m+m'}\,\mathrm{cn}(z) \\ \mathrm{dn}(z + 2Km + 2K'm'i) &= (-1)^{m'}\,\mathrm{dn}(z) \end{aligned} \tag{7.20}$$

Quarter periods:

$$\begin{aligned} \mathrm{sn}(z+K) &= \frac{\mathrm{cn}}{\mathrm{dn}}(z) & \mathrm{sn}(z+iK') &= \frac{1}{k}(z)\frac{1}{\mathrm{sn}}(z) \\ \mathrm{cn}(z+K) &= -k'\frac{\mathrm{sn}}{\mathrm{dn}}(z) & \mathrm{cn}(z+iK') &= \frac{1}{ik}\frac{\mathrm{dn}}{\mathrm{sn}}(z) \\ \mathrm{dn}(z+K) &= k'\frac{1}{\mathrm{dn}}(z) & \mathrm{dn}(z+iK') &= \frac{1}{i}\frac{\mathrm{cn}}{\mathrm{sn}}(z) \end{aligned} \tag{7.21}$$

Residues and Taylor expansions:

$$\begin{aligned} \mathrm{sn}(z) &= z - (1+k^2)\frac{z^3}{3!} + \cdots & \mathrm{sn}(iK'+z) &= \frac{1/k}{z} + \cdots \\ \mathrm{cn}(z) &= 1 - \frac{z^2}{2!} + \cdots & \mathrm{cn}(iK'+z) &= \frac{-i/k}{z} + \cdots \\ \mathrm{dn}(z) &= 1 - k^2\frac{z^2}{2!} + \cdots & \mathrm{dn}(iK'+z) &= \frac{-i}{z} + \cdots \end{aligned} \tag{7.22}$$

7.4 Complete elliptic integral of first kind $K(k)$, $K'(k)$

From (7.12) the real quarter-period K of $\mathrm{cn}(z,k)$ is

$$K(k) = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2+k^2y^2)}} \tag{7.23}$$

Changing variable $y = \cos\theta$, this is

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}} \tag{7.24}$$

the complete elliptic integral of first kind.

The imaginary quarter-period is

$$\begin{aligned} iK' &= \int_\infty^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2+k^2y^2)}} = \int_0^1 \frac{d(y^{-1})}{\sqrt{(1-y^{-2})(1-k^2+k^2y^{-2})}} \\ &= \int_0^1 \frac{idy}{\sqrt{(1-y^2)[k^2+(1-k^2)y^2]}} \\ &= iK(k') \end{aligned} \tag{7.25}$$

where

$$k^2 + k'^2 = 1 \tag{7.26}$$

7.5 $K(1/\sqrt{2})$

For $k^2 = k'^2 = \frac{1}{2}$,

$$\begin{aligned} K(1/\sqrt{2}) &= \int_0^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{1}{2}+\frac{1}{2}y^2\right)}} = \int_0^1 \frac{\sqrt{2}\,dy}{\sqrt{1-y^4}} = \int_0^1 \frac{\sqrt{2}\,d(s^{1/4})}{\sqrt{1-s}} \\ &= \frac{\sqrt{2}}{4}\int_0^1 s^{-3/4}(1-s)^{-1/2}\,ds \end{aligned} \tag{7.27}$$

The beta function is

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{7.28}$$

so

$$K(1/\sqrt{2}) = \frac{\sqrt{2}}{4}B(1/4,1/2) = \frac{\sqrt{2}}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} \tag{7.29}$$

Use

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \qquad \Gamma(1/2)^2 = \pi \qquad \Gamma(1/4)\Gamma(3/4) = \sqrt{2}\pi \tag{7.30}$$

to get

$$K(1/\sqrt{2}) = K'(1/\sqrt{2}) = \frac{\Gamma(1/4)^2}{4\pi^{1/2}} = 1.854075\ldots \tag{7.31}$$

7.6 $\langle \mathrm{cn}^2 \rangle$ for $k = 1/\sqrt{2}$

The time average $\langle \mathrm{cn}^2 \rangle$ of $\mathrm{cn}(z, k)$ over a real cycle can, by the symmetries, be calculated over a quarter-cycle

$$\langle \mathrm{cn}^2 \rangle = \frac{1}{K} \int_0^K \mathrm{cn}^2(z, k)\, dz = \frac{1}{K} \int_0^1 \frac{y^2\, dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)}} \tag{7.32}$$

For $k = 1/\sqrt{2}$,

$$\begin{aligned}
\langle \mathrm{cn}^2 \rangle &= \frac{1}{K} \int_0^1 \frac{\sqrt{2}\, y^2\, dy}{\sqrt{1 - y^4}} = \frac{1}{K} \sqrt{2} \int_0^1 \frac{s^{1/2}\, d(s^{1/4})}{\sqrt{1 - s}} \\
&= \frac{1}{K} \frac{\sqrt{2}}{4} \int_0^1 s^{-1/4} (1 - s)^{-1/2}\, ds = \frac{1}{K} \frac{\sqrt{2}}{4} B(3/4, 1/2) \\
&= \frac{1}{K} \frac{\sqrt{2}}{4} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)} = \frac{1}{K} \frac{\sqrt{2}}{4} \frac{\sqrt{2}\pi}{\Gamma(1/4)} \frac{\pi^{1/2}}{\frac{1}{4}\Gamma(1/4)} = \frac{1}{K} \frac{2\pi^{3/2}}{\Gamma(1/4)^2} \\
&= \frac{\pi}{2} \frac{1}{K^2} = 0.45694658 \ldots
\end{aligned} \tag{7.33}$$

7.7 $\langle \mathrm{cn}^2 \rangle$ for general k

For general k

$$\begin{aligned}
K \langle \mathrm{cn}^2 \rangle &= \int_0^1 \frac{y^2\, dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)}} \\
(1 - k^2) K + k^2 K \langle \mathrm{cn}^2 \rangle &= \int_0^1 \frac{(1 - k^2 + k^2 y^2)\, dy}{\sqrt{(1 - y^2)(1 - k^2 + k^2 y^2)}} \\
&= \int_0^1 \sqrt{\frac{1 - k^2 + k^2 y^2}{1 - y^2}}\, dy = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta
\end{aligned} \tag{7.34}$$

which is the complete elliptic integral of the second kind

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \tag{7.35}$$

so

$$\begin{aligned}
(1 - k^2) K + k^2 K \langle \mathrm{cn}^2 \rangle &= E \\
[3, 5.134] \qquad \langle \mathrm{cn}^2 \rangle &= \frac{E - (1 - k^2) K}{k^2 K}
\end{aligned} \tag{7.36}$$

The identity

$$[3, 8.122] \qquad E K' + E' K - K K' = \frac{\pi}{2} \tag{7.37}$$

gives a check for $k = 1/\sqrt{2}$, where $K = K'$, $E = E'$,

$$2EK - K^2 = \frac{\pi}{2} \qquad E = \frac{\pi}{4K} + \frac{1}{2} K \qquad \langle \mathrm{cn}^2 \rangle = \frac{E - \frac{1}{2} K}{\frac{1}{2} K} = \frac{\frac{\pi}{4K}}{\frac{1}{2} K} = \frac{\pi}{2K^2} \tag{7.38}$$

8 Cosmological temperature

For $E_b \gg 1$

$$k = 1/\sqrt{2} \qquad b = b_{\max} \operatorname{cn}(z,k) \qquad b_{\max} = (2E_b)^{1/4} \tag{8.1}$$

$$\langle b^2 \rangle = \langle b_{\max}^2 \operatorname{cn}^2 \rangle = (2E_b)^{1/2} \frac{\pi}{2K^2} \tag{8.2}$$

Equation (6.43) for a_{EW}

$$\frac{2}{3}\lambda^2 v^2 a_{\mathrm{EW}}^2 = 1 + \langle b^2 \rangle \tag{8.3}$$

becomes, since $\langle b^2 \rangle \gg 1$,

$$\frac{2}{3}\lambda^2 v^2 a_{\mathrm{EW}}^2 = (2E_b)^{1/2} \frac{\pi}{2K^2} \tag{8.4}$$

or

$$\begin{aligned} a_{\mathrm{EW}}^2 &= \frac{1}{\lambda^2 v^2} \frac{3}{2} (2E_b)^{1/2} \frac{\pi}{2K^2} = \frac{3\pi}{4K^2} (2E_b)^{1/2} \frac{1}{\lambda^2 v^2} \\ a_{\mathrm{EW}} &= \frac{(3\pi)^{1/2}}{2K} (2E_b)^{1/4} \frac{1}{\lambda v} = \frac{(3\pi)^{1/2}}{2K} \frac{1}{4^{1/4}\epsilon_b} \frac{\hbar}{m_{\mathrm{H}}} = \frac{(6\pi)^{1/2}}{4K\epsilon_b} \frac{\hbar}{m_{\mathrm{H}}} \end{aligned} \tag{8.5}$$

Co-moving time t is obtained by integrating

$$dt = a(T)\, dT \tag{8.6}$$

$b(T)$ is periodic in imaginary proper time T with period $K\epsilon_b$. This is very small compared to the time scale $E_{\hat{a}}^{-1/4}$ on which $a(T)$ changes by a factor of $\epsilon_{\mathrm{w}} \approx 10^{-17}$. So b is periodic in imaginary co-moving time with period $K\epsilon_b a$. The inverse temperature is the period in imaginary co-moving time

$$\frac{\hbar}{k_{\mathrm{B}} T_{\mathrm{SU(2)}}} = 4K\epsilon_b a \qquad k_{\mathrm{B}} T_{\mathrm{SU(2)}} = \frac{\hbar}{4K\epsilon_b a} = \frac{m_{\mathrm{H}}}{(6\pi)^{1/2}} \frac{a_{\mathrm{EW}}}{a} = 28.8\,\mathrm{GeV}\, \frac{a_{\mathrm{EW}}}{a} \tag{8.7}$$

$$k_{\mathrm{B}} T_{\mathrm{EW}} = 28.8\,\mathrm{GeV} \qquad T_{\mathrm{EW}} = 3.34 \times 10^{14}\,\mathrm{K} \tag{8.8}$$

9 Solution of the $\hat{a}$ oscillator

9.1 Solution for $\hat{a}$ in co-moving time

Co-moving time t is given by

$$dt = a(T) dT \qquad a^2(-dT^2 + ds_{S^3}^2) = -dt^2 + a^2 ds_{S^3}^2 \tag{9.1}$$

Define the dimensionless co-moving time $\hat{t}$

$$t = t_{\mathrm{I}} \hat{t} \qquad d\hat{t} = \hat{a}(T) dT \tag{9.2}$$

The $\hat{a}$ energy equation (4.27)

$$\frac{1}{2}(\partial_T \hat{a})^2 + \frac{1}{2}\hat{a}^2 - \frac{1}{2}\hat{a}^4 = E_{\hat{a}} \tag{9.3}$$

becomes

$$\frac{1}{2}\hat{a}^2(\partial_{\hat{t}} \hat{a})^2 + \frac{1}{2}\hat{a}^2 - \frac{1}{2}\hat{a}^4 = E_{\hat{a}} \tag{9.4}$$

Change variables,

$$\hat{a} = (2E_{\hat{a}})^{1/4} u \qquad \epsilon_a = (2E_{\hat{a}})^{-1/4} \tag{9.5}$$

The energy equation now becomes

$$(2E_{\hat{a}})\frac{1}{2}u^2(\partial_{\hat{t}}u)^2 + (2E_{\hat{a}})^{1/2}\frac{1}{2}u^2 - (2E_{\hat{a}})\frac{1}{2}u^4 = E_{\hat{a}}$$
$$u^2(\partial_{\hat{t}}u)^2 + (2E_{\hat{a}})^{-1/2}u^2 - u^4 = 1 \tag{9.6}$$
$$u^2(\partial_{\hat{t}}u)^2 + \epsilon_a^2 u^2 - u^4 = 1$$

Change variable again

$$u^2 = w + \frac{1}{2}\epsilon_a^2 \qquad \frac{1}{4}(\partial_{\hat{t}}w)^2 + \epsilon_a^2\left(w + \frac{1}{2}\epsilon_a^2\right) - \left(w + \frac{1}{2}\epsilon_a^2\right)^2 = 1$$
$$\frac{1}{4}(\partial_{\hat{t}}w)^2 - w^2 = 1 - \frac{1}{4}\epsilon_a^4 \tag{9.7}$$

Take the $\hat{t}$ derivative

$$\frac{1}{2}(\partial_{\hat{t}}w)\partial_{\hat{t}}^2 w - 2w(\partial_{\hat{t}}w) = 0$$
$$\partial_{\hat{t}}^2 w - 4w = 0 \tag{9.8}$$
$$w = A_1 e^{2\hat{t}} + A_2 e^{-2\hat{t}}$$

Fix the origin in $\hat{t}$ by

$$u(0) = 0 \qquad A_1 + A_2 = -\frac{1}{2}\epsilon_a^2 \tag{9.9}$$

Substitute in (9.7)

$$\left(A_1 e^{2\hat{t}} - A_2 e^{-2\hat{t}}\right)^2 - \left(A_1 e^{2\hat{t}} + A_2 e^{-2\hat{t}}\right)^2 = 1 - \frac{1}{4}\epsilon_a^4$$
$$-4A_1A_2 = 1 - \frac{1}{4}\epsilon_a^4$$
$$(A_1 - A_2)^2 = (A_1 + A_2)^2 - 4A_1A_2 = \frac{1}{4}\epsilon_a^4 + 1 - \frac{1}{4}\epsilon_a^4 = 1 \qquad A_1 - A_2 = \pm 1 \tag{9.10}$$
$$A_1 = \pm\frac{1}{2} - \frac{1}{4}\epsilon_a^2 \qquad A_2 = \mp\frac{1}{2} - \frac{1}{4}\epsilon_a^2$$
$$u^2 = \left(\pm\frac{1}{2} - \frac{1}{4}\epsilon_a^2\right)e^{2\hat{t}} + \left(\mp\frac{1}{2} - \frac{1}{4}\epsilon_a^2\right)e^{-2\hat{t}} + \frac{1}{2}\epsilon_a^2 \tag{9.11}$$

Use the solution with u increasing with time

$$u^2 = \left(\frac{1}{2} - \frac{1}{4}\epsilon_a^2\right)e^{2\hat{t}} + \left(-\frac{1}{2} - \frac{1}{4}\epsilon_a^2\right)e^{-2\hat{t}} + \frac{1}{2}\epsilon_a^2 \tag{9.12}$$

Define

$$k_a^2 = \frac{1}{2} + \frac{1}{4}\epsilon_a^2 \tag{9.13}$$

Then

$$u^2 = \left(1 - k_a^2\right)e^{2\hat{t}} - k_a^2 e^{-2\hat{t}} + 2k_a^2 - 1 = (e^{2\hat{t}} - 1)(1 - k_a^2 + k_a^2 e^{-2\hat{t}}) \tag{9.14}$$

So the solution for $\hat{a}$ is

$$\epsilon_a = (2E_{\hat{a}})^{-1/4} \qquad k_a^2 = \frac{1}{2} + \frac{1}{4}\epsilon_a^2$$
$$\hat{a} = (2E_{\hat{a}})^{1/4}u \qquad u = \sqrt{\left(e^{2\hat{t}} - 1\right)\left(1 - k_a^2 + k_a^2 e^{-2\hat{t}}\right)} \tag{9.15}$$

Proper time is given by

$$\begin{gathered}\frac{d\hat{t}}{dT} = \hat{a} = \epsilon_a^{-1}\sqrt{\left(e^{2\hat{t}}-1\right)\left(1-k_a^2+k_a^2e^{-2\hat{t}}\right)} \\ \frac{e^{-\hat{t}}}{\epsilon_a^{-1}}\frac{d\hat{t}}{dT} = \sqrt{\left(1-e^{-2\hat{t}}\right)\left(1-k_a^2+k_a^2e^{-2\hat{t}}\right)} \\ T=\epsilon_a z_a \qquad y_a = e^{-\hat{t}} \\ -\frac{dy_a}{dz_a} = \sqrt{(1-y_a^2)\,(1-k_a^2+k_a^2y_a^2)} \\ y_a = \mathrm{cn}(z_a,k_a)\end{gathered} \tag{9.16}$$

so the relation between co-moving and proper time is

$$e^{-\hat{t}} = \mathrm{cn}(z_a,k_a) \qquad T=\epsilon_a z_a \tag{9.17}$$

Since $E_{\hat{a}} \gg 1$,

$$\begin{gathered}k_a^2 = \frac{1}{2} \qquad \hat{a} = (2E_{\hat{a}})^{1/4}u \qquad u = \sqrt{\left(e^{2\hat{t}}-1\right)\left(\frac{1}{2}+\frac{1}{2}e^{-2\hat{t}}\right)} = \sqrt{\sinh\left(2\hat{t}\right)} \\ K_a = K(1/\sqrt{2}) = \frac{\Gamma(1/4)^2}{4\pi^{1/2}} = 1.854075\ldots\end{gathered} \tag{9.18}$$

The ratio of the b oscillation time scale to the expansion time scale is

$$\frac{\epsilon_b}{\epsilon_a} = \frac{(8E_b)^{-1/4}}{(2E_{\hat{a}})^{-1/4}} = \frac{\epsilon_{\mathrm{w}}}{\sqrt{2}} = 2.40\times10^{-17} \tag{9.19}$$

9.2 $\hat{a}_{\mathrm{EW}}$

As t goes from 0 to ∞,

t	$\hat{t}$	u	$\hat{a}$	a	$e^{-\hat{t}} = \mathrm{cn}(z_a,k_a)$	$T=\epsilon_a z_a$
0	0	0	0	0	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t_{EW}	$\hat{t}_{\mathrm{EW}}$	u_{EW}	$\hat{a}_{\mathrm{EW}}$	a_{EW}	$e^{-\hat{t}_{\mathrm{EW}}}$	T_{EW}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	∞	∞	∞	∞	0	$K_a\epsilon_a$

(9.20)

Here T_{EW} is the proper time at the onset of the electroweak transition, not the temperature. Equation (8.5) gives

$$a_{\mathrm{EW}} = \frac{(3\pi)^{1/2}}{2K}\frac{(2E_b)^{1/4}}{\lambda v} \qquad a_{\mathrm{EW}}^2 = \frac{3\pi}{4K^2}\frac{(2E_b)^{1/4}}{\lambda^2 v^2} \tag{9.21}$$

Recall (5.11), (5.12)

$$\begin{gathered}a^2 = t_{\mathrm{I}}^2\hat{a}^2 \qquad t_{\mathrm{I}}^2 = \frac{24}{\kappa\hbar\lambda^2v^4} = (2.55\times10^{-10}\,\mathrm{s})^2 \\ E_b = \epsilon_{\mathrm{w}}^{-4}E_{\hat{a}} \qquad \epsilon_{\mathrm{w}}^2 = \frac{\kappa\hbar\lambda v^2}{4\cdot3^{1/2}g} = (3.39\times10^{-17})^2\end{gathered} \tag{9.22}$$

So the dimensionless scale factor $\hat{a}_{\rm EW}$ at the onset of the electroweak transition is

$$\begin{aligned}
\hat{a}_{\rm EW}^2 = t_{\rm I}^{-2} a_{\rm EW}^2 &= \frac{\kappa\hbar\lambda^2 v^4}{24}\frac{3\pi}{4K^2}\frac{(2E_b)^{1/4}}{\lambda^2 v^2} = \frac{\pi}{32K^2}\kappa\hbar v^2 (2\epsilon_{\rm w}^{-4}E_{\hat{a}})^{1/2} \\
&= \frac{\pi}{32K^2}\kappa\hbar v^2 (2E_{\hat{a}})^{1/2}\frac{4\cdot 3^{1/2} g}{\kappa\hbar\lambda v^2} \\
&= \frac{3^{1/2}\pi}{8K^2}\frac{g}{\lambda}(2E_{\hat{a}})^{1/2} = \frac{3^{1/2}\pi}{8K^2}\frac{2m_W}{m_{\rm H}}(2E_{\hat{a}})^{1/2} = 0.254\,(2E_{\hat{a}})^{1/2}
\end{aligned} \tag{9.23}$$

In terms of the solution (9.18) for $\hat{a}(t)$,

$$\begin{aligned}
u_{\rm EW}^2 &= (2E_{\hat{a}})^{-1/2}\hat{a}_{\rm EW}^2 = \frac{3^{1/2}\pi}{4K^2}\frac{m_W}{m_{\rm H}} = 0.254 \\
\hat{t}_{\rm EW} &= \frac{1}{2}\,\text{arcsinh}(u_{\rm EW}^2) = \frac{1}{2}\,\text{arcsinh}(0.254) = 0.126 \\
t_{\rm EW} &= \hat{t}_{\rm EW} t_{\rm I} = 0.126\, t_{\rm I} = 3.21\times 10^{-11}\,\text{s} \\
T_{\rm EW} &= \text{arccn}(e^{-\hat{t}_{\rm EW}}, k_a)\epsilon_a = 0.501\,\epsilon_a
\end{aligned} \tag{9.24}$$

9.3 The function $u(z,k)$

This subsection is not used in the paper. In this subsection, write z in place of z_a and k in place of k_a.

The function $u(z,k)$ given by

$$u = \sqrt{\left(e^{2\hat{t}}-1\right)\left(1-k^2+k^2 e^{-2\hat{t}}\right)} \qquad e^{-\hat{t}} = \text{cn}(z,k) \tag{9.25}$$

has some nice properties. The identities (7.16) for the Jacobi elliptic functions give

$$\begin{aligned}
1-e^{-2\hat{t}} &= 1-\text{cn}^2(z,k) = \text{sn}^2(z,k) \\
1-k^2+k^2 e^{-2\hat{t}} &= 1-k^2+k^2\,\text{cn}^2(z,k) = \text{dn}^2(z,k)
\end{aligned} \tag{9.26}$$

so

$$u(z,k) = \frac{\text{sn}\,\text{dn}}{\text{cn}}(z,k) \tag{9.27}$$

Then identity (7.17) gives

$$u(z,k) = \sqrt{\frac{1-\text{cn}(2z,k)}{1+\text{cn}(2z,k)}} \tag{9.28}$$

and by identity (7.14)

$$u(z,k) = \frac{d}{dz}\left[-\ln\text{cn}(z,k)\right] = \frac{d\hat{t}}{dz} \tag{9.29}$$

(which returns to the starting point $\hat{a} = d\hat{t}/dT$.)

So $u(z,k)$ has the equivalent forms

$$u(z,k) = \frac{\text{sn}\,\text{dn}}{\text{cn}}(z,k) = \sqrt{\frac{1-\text{cn}(2z,k)}{1+\text{cn}(2z,k)}} = \frac{d}{dz}\left[-\ln\text{cn}(z,k)\right] \tag{9.30}$$

From the quarter-period identities (7.21), $u(z) = u(z,k)$ satisfies

$$u(z+K) = \frac{-1}{u(z)} \qquad u(z+iK') = \frac{1}{u(z)} \qquad u(-z) = -u(z) \tag{9.31}$$

from which

$$u(z) = u(z+2K) = u(z+2iK') = -u(z+K+iK') \tag{9.32}$$

From the expansions (7.22),

$$u(z-K) = \frac{-1}{z} + O(z) \qquad u(z) = z + O(z^3) \qquad u(z+K) = \frac{-1}{z} + O(z) \tag{9.33}$$

Equation (7.19) lists the poles and zeros of cn, sn, and dn,

$$\begin{array}{c|c|c} & \text{pole} & \text{zero} \\ \hline \text{sn} & iK' & 0 \\ \text{cn} & iK' & K \\ \text{dn} & iK' & K+iK' \end{array} \qquad + 2mK + 2inK' \tag{9.34}$$

So the zeros and poles of $u = \operatorname{sn}\operatorname{dn} / \operatorname{cn}$ are the lattice points $nK + n'iK'$, for all integers n, n'. The zeros are at the lattice points with $n + n'$ even, the poles at the points with $n + n'$ odd. In particular, $u(z)$ has poles at $z = \pm K$ and $\pm iK'$ and zeros at $z = 0$ and $K + iK'$.

$u(z,k)$ satisfies the differential equation

$$\left(\frac{du}{dz}\right)^2 = (u^2+1)^2 - 4k^2u^2 \tag{9.35}$$

derived by

$$\begin{aligned} u^2 = \frac{\operatorname{sn}^2\operatorname{dn}^2}{\operatorname{cn}^2} = \frac{(1-\operatorname{cn}^2)(k^2\operatorname{cn}^2 - k^2 + 1)}{\operatorname{cn}^2} &= -k^2\operatorname{cn}^2 + 2k^2 - 1 - (k^2-1)\operatorname{cn}^{-2} \\ 2uu' &= \left[-2k^2\operatorname{cn} + 2(k^2-1)\operatorname{cn}^{-3}\right](-\operatorname{sn}\operatorname{dn}) \\ u' &= k^2\operatorname{cn}^2 - (k^2-1)\operatorname{cn}^{-2} \\ u' - u^2 - 1 = 2k^2(\operatorname{cn}^2 - 1) \qquad & u' + u^2 + 1 = 2k^2 + 2(-k^2+1)\operatorname{cn}^{-2} \\ (u'-u^2-1)(u'+u^2+1) &= -4k^2u^2 \\ u'^2 &= (u^2+1)^2 - 4k^2u^2 \end{aligned} \tag{9.36}$$

9.4 Direct solution of the $\hat{a}$ oscillator by an elliptic function

This subsection is not used in the paper. The solution for $\hat{a}(T)$ is found directly, without going to co-moving time.

Change variables from T and $\hat{a}(T)$ to z_a and $u(z_a)$

$$T = \epsilon_a z_a \qquad \hat{a} = \epsilon_a^{-1} u \qquad \epsilon_a = (2E_{\hat{a}})^{-1/4} \tag{9.37}$$

The energy equation (4.27)

$$E_{\hat{a}} = \frac{1}{2}(\partial_T \hat{a})^2 + \frac{1}{2}\hat{a}^2 - \frac{1}{2}\hat{a}^4 \tag{9.38}$$

becomes

$$\begin{aligned}(\partial_T \hat{a})^2 &= 2E_{\hat{a}} - \hat{a}^2 + \hat{a}^4 \\ \epsilon_a^{-2}\epsilon_a^{-2}(\partial_{z_a} u)^2 &= \epsilon_a^{-4} - \epsilon_a^{-2}u^2 + \epsilon_a^{-4}u^4 \\ (\partial_{z_a} u)^2 &= 1 - \epsilon_a^2 u^2 + u^4\end{aligned} \tag{9.39}$$

Define

$$k_a^2 = \frac{1}{2} + \frac{1}{4}\epsilon_a^2 \tag{9.40}$$

so

$$\begin{aligned}(\partial_{z_a} u)^2 &= 1 - 4\left(k_a^2 - \frac{1}{2}\right)u^2 + u^4 \\ &= (1+u^2)^2 - 4k_a^2 u^2\end{aligned} \tag{9.41}$$

Let

$$u = \sqrt{\frac{1-y_a}{1+y_a}} \qquad u^2 = \frac{1-y_a}{1+y_a} = \frac{2}{1+y_a} - 1 \tag{9.42}$$

Then

$$\begin{aligned}\left[\frac{d(u^2)}{dz_a}\right]^2 &= \left(2u\frac{du}{dz_a}\right)^2 \\ \left[\frac{-2}{(1+y_a)^2}\frac{dy_a}{dz_a}\right]^2 &= 4u^2\left[(1+u^2)^2 - 4k_a^2u^2\right] \\ \frac{4}{(1+y_a)^4}\left(\frac{dy_a}{dz_a}\right)^2 &= 4\left(\frac{1-y_a}{1+y_a}\right)\left[\frac{4}{(1+y_a)^2} - 4k_a^2\left(\frac{1-y_a}{1+y_a}\right)\right] \\ \frac{1}{4}\left(\frac{dy_a}{dz_a}\right)^2 &= (1+y_a)^4\left(\frac{1-y_a}{1+y_a}\right)\left[\frac{1}{(1+y_a)^2} - k_a^2\left(\frac{1-y_a}{1+y_a}\right)\right] \\ \frac{1}{4}\left(\frac{dy_a}{dz_a}\right)^2 &= (1+y_a)(1-y_a)\left[1 - k_a^2(1+y_a)(1-y_a)\right] \\ \frac{1}{4}\left(\frac{dy_a}{dz_a}\right)^2 &= (1-y_a^2)\left(1 - k_a^2 + k_a^2 y_a^2)\right)\end{aligned} \tag{9.43}$$

So

$$y_a = \mathrm{cn}(2z_a, k_a) \qquad u(z_a, k_a) = \sqrt{\frac{1-\mathrm{cn}(2z_a,k_a)}{1+\mathrm{cn}(2z_a,k_a)}} \tag{9.44}$$

$$\begin{gathered}a = (2E_{\hat{a}})^{1/4}u(z_a,k_a) \qquad T = \epsilon_a z_a \qquad \epsilon_a = (2E_{\hat{a}})^{-1/4} \qquad k_a^2 = \frac{1}{2} + \frac{1}{4}\epsilon_a^2 \\ K_a = K(k_a) \qquad K_a' = K'(ka)\end{gathered} \tag{9.45}$$

Numerical calculations for "Origins of Cosmological Temperature"

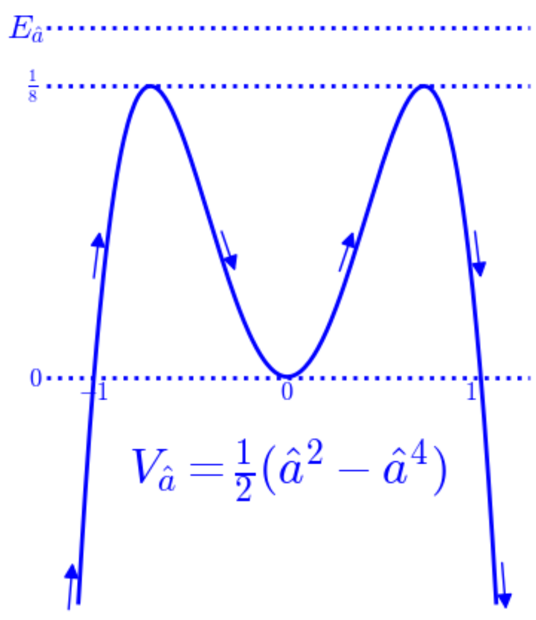
This SageMath notebook performs numerical calculations for the paper *Origins of Cosmological Temperature* and the supplemental note *Calculations for "Origins of Cosmological Temperature"*.

It makes graphs of the two anharmonic potentials and does some arithmetic.

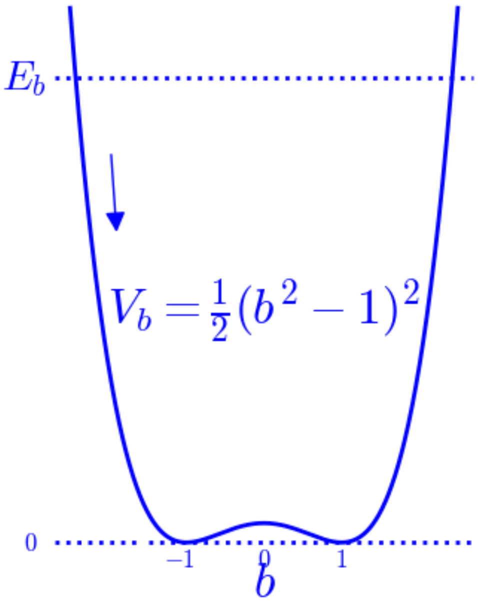
Section headings are as in the supplemental note.

3.5 Action in dimensionless variables

In [1]:
```
ah = var('ah')
E_ah = 0.15
ah_lim=1.08
ah_plot=plot((1/2)*(ah^2-ah^4), (ah,-ah_lim,ah_lim),aspect_ratio=12)
ah_plot+=text(r'$E_{\hat{a}}$', (-1.35,E_ah),fontsize=8)
ah_plot+=text(r'$0$',(-1.3,0),fontsize=6)
ah_plot+=text(r'$\frac{1}{8}$',(-1.32,0.125),fontsize=6,aspect_ratio=1)
ah_plot+=text(r'$0$',(0,-0.006),fontsize=6,aspect_ratio=1)
ah_plot+=text(r'$1$',(0.95,-0.006),fontsize=6,aspect_ratio=1)
ah_plot+=text(r'$-1$',(-1.0,-0.006),fontsize=6,aspect_ratio=1)
ah_plot+=text(r'$V_{\hat{a}}=\frac{1}{2}(\hat{a}^2-\hat{a}^4)$',(0,-0.04),fontsize=12)
ah_plot+= plot(0.125,(ah,-1.25,1.25),linestyle=":")
ah_plot+= plot(0,(ah,-1.25,1.25),linestyle=":")
ah_plot+= plot(E_ah,(ah,-1.25,1.25),linestyle=":")
ah_plot+=arrow((1.11,-0.08),(1.13,-0.08-.02),width=.5,arrowsize=1.5)
ah_plot+=arrow((-1.13,-0.08-.02),(-1.11,-0.08),width=.5,arrowsize=1.5)
ah_plot+=arrow((0.97,1/16),(1.00,1/16-.02),width=.5,arrowsize=1.5)
ah_plot+=arrow((-1.00,1/16-.02),(-0.97,1/16),width=.5,arrowsize=1.5)
ah_plot+=arrow((-.34,1/16),(-.27,1/16-.017),width=.5,arrowsize=1.5)
ah_plot+=arrow((.27,1/16-.017),(.34,1/16),width=.5,arrowsize=1.5)
ah_plot.save('plot_ah.pdf',dpi=200,axes=False)
show(ah_plot,axes=False,dpi=200,figsize=[3.2,2.4])
```



```
In [2]: b = var('b')
        E_b = 12
        b_plot=plot((1/2)*(b^2-1)^2, (b,-2.5,2.5),aspect_ratio=.5)
        b_plot+=text(r'$E_b$',(-3.1,E_b),fontsize=10)
        b_plot+=text(r'$0$',(-3.0,0),fontsize=6)
        b_plot+=text(r'$V_b=\frac{1}{2}(b^2-1)^2$',(0,E_b/2),fontsize=12)
        b_plot+=text(r'$b$',(0,-1),fontsize=12)
        b_plot+= plot(0,(b,-2.7,2.7),linestyle=":")
        b_plot+= plot(E_b,(b,-2.7,2.7),linestyle=":")
        b_plot+=text(r'$1$',(1.0,-0.4),fontsize=6)
        b_plot+=text(r'$0$',(0.0,-0.4),fontsize=6)
        b_plot+=text(r'$-1$',(-1.09,-0.4),fontsize=6)
        b_plot+=arrow((-1.97,E_b-2),(-1.9,E_b-4),width=.5,arrowsize=2)
        b_plot.save('plot_b.pdf',dpi=200,axes=False)
        show(b_plot,axes=False,dpi=200,figsize=[3.2,2.4])
```



5.1 Fundamental constants

```
In [3]: %display latex
        LE = lambda latex_string: LatexExpr(latex_string);
```

declare units as variables

```
In [4]: s = var('s', domain='positive'); assume(s,'real');
        GeV = var('GeV', domain='positive'); assume(GeV,'real');
        J = var('J', domain='positive'); assume(J,'real');
        m = var('m', domain='positive'); assume(m,'real');
        meters = var('meters', domain='positive'); assume(meters,'real');
        kg = var('kg', domain='positive'); assume(kg,'real');
        K = var('K', domain='positive'); assume(K,'real');
        C = var('C', domain='positive'); assume(C,'real');
```

fundamental constents from NIST 2018

In [5]:
```
c = 299792458 * meters * s^(-1);
e_charge = 1.602176634  * 10^(-19) * C;
hbar = 1.054571817*10^(-34)*J*s;
kB = 1.380649*10^(-23)*J*K^(-1);
G = 6.67430*10^(-11)*m^3*kg^(-1)* s^(-2);
kappa = N(8*pi)*G;
#
pretty_print(LE(r"c ="),c);
pretty_print(LE(r"e ="),e_charge);
pretty_print(LE(r"\hbar ="),hbar);
pretty_print(LE(r"k_{B} ="),kB);
pretty_print(LE(r"G ="),G);
pretty_print(LE(r"\kappa ="),kappa);
```

$$c = \frac{299792458\, meters}{s}$$

$$e = \left(1.60217663400000 \times 10^{-19}\right)\ C$$

$$\hbar = \left(1.05457181700000 \times 10^{-34}\right)\ Js$$

$$k_B = \frac{\left(1.38064900000000 \times 10^{-23}\right)\ J}{K}$$

$$G = \frac{\left(6.67430000000000 \times 10^{-11}\right)\ m^3}{kgs^2}$$

$$\kappa = \frac{\left(1.67743454782835 \times 10^{-9}\right)\ m^3}{kgs^2}$$

use c=1 units with unit of energy = GeV

In [6]:
```
m = c^(-1)*meters;
J = e_charge^(-1) * C * 10^(-9) * GeV
kg = J*s^2*m^(-2)
def conv(*args):
    return [arg.subs(m=m,kg=kg,J=J) for arg in args]
[hbar,kB,G,kappa] = conv(hbar,kB,G,kappa)
pretty_print(LE(r"\hbar ="),hbar);
pretty_print(LE(r"k_{B} ="),kB);
pretty_print(LE(r"G ="),G);
pretty_print(LE(r"\kappa ="),kappa);
```

$$\hbar = \left(6.58211956547607 \times 10^{-25}\right)\ GeVs$$

$$k_B = \frac{\left(8.61733326214518 \times 10^{-14}\right)\ GeV}{K}$$

$$G = \frac{\left(4.41583261432942 \times 10^{-63}\right)\ s}{GeV}$$

$$\kappa = \frac{\left(1.10981978405276 \times 10^{-61}\right)\ s}{GeV}$$

5.2 Standard Model coupling constants from PDG (2018, 2019)

In [7]:
```
GFermi =  1.1663787*10^(-5)*GeV^(-2);
mW = 80.379*GeV;
mH = 125.10*GeV;
#
pretty_print(LE(r"G_F ="),GFermi);
pretty_print(LE(r"m_W ="),mW);
pretty_print(LE(r"m_H ="),mH);
```

$$G_F = \frac{0.0000116637870000000}{GeV^2}$$

$$m_W = 80.3790000000000\ GeV$$

$$m_H = 125.100000000000\ GeV$$

In [8]:
```
hbar_v = N(2^(-1/4))*GFermi^(-1/2)
pretty_print(LE(r"\hbar v  = 2^{-1/4} G_F^{-1/2}="),hbar_v)
v = hbar_v/hbar
pretty_print(LE(r"v ^{-1} ="),1/v);
g = 2*mW/hbar_v;
pretty_print(LE(r"g = \frac{2 m_W}{\hbar v }="),g)
lambdaH = mH/hbar_v;
pretty_print(LE(r"\lambda = \frac{m_H}{\hbar v }="),lambdaH)
```

$$\hbar v = 2^{-1/4} G_F^{-1/2} = 246.219650794137\, GeV$$

$$v^{-1} = \left(2.67327142421274 \times 10^{-27}\right)\, s$$

$$g = \frac{2m_W}{\hbar v} = 0.652904833068782$$

$$\lambda = \frac{m_H}{\hbar v} = 0.508082923505546$$

5.3 Gravitational and weak time scales t_{grav}, t_{W}

In [9]:
```
tgrav = sqrt(kappa*hbar);
pretty_print(LE(r"t_{\mathrm{grav}} = (\hbar\kappa)^{1/2}=\
(8\pi)^{1/2} t_{P}="),N((8*pi)^(1/2)),LE(r"t_{P}\\="),tgrav)
```

$$t_{\mathrm{grav}} = (\hbar\kappa)^{1/2} = (8\pi)^{1/2} t_P = 5.01325654926200 t_P$$
$$= \left(2.70277015574135 \times 10^{-43}\right)\, s$$

In [10]:
```
tW = hbar/mW;
pretty_print(LE(r"t_{\mathrm{W}}=\frac{\hbar}{m_W}="),tW)
```

$$t_{\mathrm{W}} = \frac{\hbar}{m_W} = \left(8.18885475743176 \times 10^{-27}\right)\, s$$

5.4 The scalar field energy density $\mathcal{E}_0$

In [11]:
```
E0 = hbar*lambdaH^2*v^4/8;
ratio1 = tW^4*E0/hbar;
pretty_print(LE(r"\frac{1}{\hbar}\mathcal{E}_{0}="),\
             ratio1,LE(r"\: t_{\mathrm{W}}^{-4}"))
```

$$\frac{1}{\hbar}\mathcal{E}_0 = 2.84118595562545\, t_{\mathrm{W}}^{-4}$$

5.5 Seesaw time scale t_I

In [12]:
```
tI = (3/(kappa*E0))^(1/2);
ratio2 = tI*tgrav/tW^2;
pretty_print(LE(r"t_{I}="),
             ratio2,LE(r"\:\frac{t_{\mathrm{W}}^{2}}{t_{\mathrm{grav}}}"),\
            LE(r"="),tI)
```

$$t_I = 1.02756853595816\, \frac{t_{\mathrm{W}}^2}{t_{\mathrm{grav}}} = \left(2.54945892615748 \times 10^{-10}\right)\, s$$

In [13]:
```
ratio12 = N(sqrt(3/2))*g^2/lambdaH;
pretty_print(LE(r"\sqrt{\frac32}\frac{g^2}{\lambda}="),ratio12);
```

$$\sqrt{\frac{3}{2}}\frac{g^2}{\lambda} = 1.02756853595816$$

In [14]:
```
tI*c;
pretty_print(LE(r"t_{I} c="),tI*c);
```

$$t_I c = 0.0764308558042792\, meters$$

5.6 Seesaw ratio ϵ_W

In [15]:
```
epsilonW = (kappa*hbar/(2*g^2*tI^2))^(1/4);
pretty_print(LE(r"\epsilon_{W}="),epsilonW)
```

$$\epsilon_W = 3.38842679174089 \times 10^{-17}$$

In [16]:
```
ratio3 = epsilonW*tW/tgrav;
ratio4 = epsilonW*tI/tW;
ratio34 = sqrt(ratio3*ratio4);
pretty_print(LE(r"\epsilon_{W}="),ratio34,
             LE(r"\:\left(\frac{t_{\mathrm{grav}}}{t_{I}}\right)^{1/2}"))
pretty_print(LE(r"\epsilon_{W}="),ratio3,
             LE(r"\:\frac{t_{\mathrm{grav}}}{t_{\mathrm{W}}}"))
pretty_print(LE(r"\epsilon_{W}="),ratio4,
             LE(r"\:\frac{t_{\mathrm{W}}}{t_{I}}"))
```

$$\epsilon_W = 1.04068084127939 \left(\frac{t_{\mathrm{grav}}}{t_I}\right)^{1/2}$$

$$\epsilon_W = 1.02662576744880 \frac{t_{\mathrm{grav}}}{t_{\mathrm{W}}}$$

$$\epsilon_W = 1.05492833683428 \frac{t_{\mathrm{W}}}{t_I}$$

In [17]:
```
ratio134=(2*g^2)^(-1/4);
pretty_print(LE(r"\left(\frac{1}{2g^2}\right)^{1/4}="),ratio134)
```

$$\left(\frac{1}{2g^2}\right)^{1/4} = 1.04068084127939$$

In [18]:
```
ratio13=(lambdaH^2/(3*g^6))^(1/4);
pretty_print(LE(r"\left(\frac{\lambda^2}{3g^6}\right)^{1/4}="),ratio13)
```

$$\left(\frac{\lambda^2}{3g^6}\right)^{1/4} = 1.02662576744880$$

In [19]:
```
ratio14=((3*g^2)/(4*lambdaH^2))^(1/4);
pretty_print(LE(r"\left(\frac{3g^2}{4\lambda^2}\right)^{1/4}="),ratio14)
```

$$\left(\frac{3g^2}{4\lambda^2}\right)^{1/4} = 1.05492833683428$$

5.7 Units of action for the two oscillators

In [20]:
```
ratio5=6*N(pi)^2/g^2;
pretty_print(LE(r"\frac{6\pi^{2}}{g^2}="),ratio5)
```

$$\frac{6\pi^2}{g^2} = 138.915667118982$$

7.5 $K(1/\sqrt{2})$

In [21]:
```
K = N(gamma(1/4)^2/(4*pi^(1/2)));
pretty_print(LE(r"K(1/\sqrt{2}) = \frac{\Gamma(1/4)^{2}}{4 \pi^{1/2}}="),K)
```

$$K(1/\sqrt{2}) = \frac{\Gamma(1/4)^2}{4\pi^{1/2}} = 1.85407467730137$$

7.6 $\langle \mathrm{cn}^2 \rangle$ for $k = 1/\sqrt{2}$

In [22]:
```
cn2ave = N(pi/2)/K^2;
pretty_print(LE(r"\langle \mathrm{cn}^2 \rangle=\frac{2}{\pi^{2}}\frac{1}{K^{2}}="),cn2ave)
```

$$\langle \mathrm{cn}^2 \rangle=\frac{2}{\pi^{2}}\frac{1}{K^{2}}= 0.456946581044464$$

8. Cosmological temperature

In [23]:
```
kT = mH/N((6*pi)^(1/2));
pretty_print(LE(r"k_B T ="),kT)
```

$k_B T = 28.8142120659094\ GeV$

In [24]:
```
pretty_print(LE(r"T ="),kT/kB)
```

$T = 3.34375046077032 \times 10^{14}\ K$

9.1 Solution for $\hat{a}$ in co-moving time

In [25]:
```
pretty_print(LE(r"\frac{\epsilon_W}{\sqrt{2}} ="),epsilonW/N(sqrt(2)))
```

$$\frac{\epsilon_W}{\sqrt{2}} = 2.39597956199416 \times 10^{-17}$$

9.2 $\hat{a}_{\mathrm{EW}}$

In [26]:
```
ratio6 =N(3^(1/2)*pi/(8*K^2))*(2*mW/mH);
pretty_print(LE(r"\hat a^2_{\mathrm{EW}} ="),\
             LE(r"\frac{3^{1/2}\pi}{8 K^2}\frac{2m_W}{m_H}(2 E_{\hat a})^{1/2}="),\
             ratio6,LE(r"\:(2 E_{\hat a})^{1/2}"))
```

$$\hat a^2_{\mathrm{EW}} = \frac{3^{1/2}\pi}{8 K^2}\frac{2m_W}{m_H}(2 E_{\hat a})^{1/2}= 0.254261938075174\ (2 E_{\hat a})^{1/2}$$

In [27]:
```
pretty_print(LE(r"\hat a_{\mathrm{EW}} ="),\
             sqrt(ratio6),\
            LE(r"\:(2 E_{\hat a})^{1/4}"))
```

$\hat a_{\mathrm{EW}} = 0.504243927157457\ (2 E_{\hat a})^{1/4}$

In [28]:
```
thatEW = asinh(ratio6)/2;
pretty_print(LE(r"\hat t_{\mathrm{EW}} ="),thatEW)
```

$\hat t_{\mathrm{EW}} = 0.125799532989201$

In [29]:
```
tEW = thatEW * tI;
pretty_print(LE(r"t_{\mathrm{EW}} ="),tEW)
```

$t_{\mathrm{EW}} = \left(3.20720742285761 \times 10^{-11}\right)\ s$

In [30]:
```
TEW = inverse_jacobi('cn', e^(-thatEW), 0.5);
pretty_print(LE(r"T_{\mathrm{EW}} ="),TEW,LE(r"\;\epsilon_a"))
```

$T_{\mathrm{EW}} = 0.501068706214232\ \epsilon_a$

References

[1] E. Tiesinga *et al.*, "The 2018 CODATA Recommended Values of the Fundamental Physical Constants." `http://physics.nist.gov/constants`.

[2] **Particle Data Group** Collaboration, M. Tanabashi *et al.*, "Review of Particle Physics," *Phys. Rev. D* **98** no. 3, (2018) 030001. and 2019 update, `http://pdg.lbl.gov/`, Sections 1 and 10.

[3] I. Gradshteyn, A. Jeffrey, and I. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, 1996. Sections 8.11, 8.14, 8.15.

[4] F. W. J. Olver *et al.*, "*NIST Digital Library of Mathematical Functions*." Release 1.0.26 of 2020-03-15, `http://dlmf.nist.gov/`. Chapter 22 and section 19.2.