Known

Definition.
$$\mu\hat{}(G,\{k,l\}) = \max\{|A| \ : \ A \subseteq G, (k\hat{}A) \cap (l\hat{}A) = \emptyset\}.$$

Corollary (G.14)**.** For every group G of order n and exponent κ we have
$$\mu(G,\{k,l\}) \geq \mu(\mathbb{Z}_\kappa,\{k,l\}) \cdot \frac{n}{\kappa}.$$

Theorem (G.18)**.** Let κ be the exponent of G. Then
$$\mu(G,\{2,1\}) = \mu(\mathbb{Z}_\kappa,\{2,1\}) \cdot \frac{n}{\kappa} = v_1(\kappa,3) \cdot \frac{n}{\kappa}.$$

Proposition (G.63)**.** Suppose that G is an abelian group of order n and exponent κ. Then, for all positive integers k and l with $k > l$ we have
$$\mu\hat{}(G,\{k,l\}) \geq \mu(G,\{k,l\}) \geq v_{k-l}(\kappa, k+l) \cdot \frac{n}{\kappa}.$$

Corollary (G.65)**.** For positive integers n, k, and l with $l < k \leq n$ we have
$$\mu\hat{}(\mathbb{Z}_n,\{k,l\}) \geq \left\lfloor \frac{n + k^2 + l^2 - \gcd(n, k-l) - 1}{k+l} \right\rfloor.$$

Theorem (G.67)**.** (Zannier; cf. [205]) For all positive integers we have
$$\mu\hat{}(\mathbb{Z}_n,\{2,1\}) = \begin{cases} \left(1 + \frac{1}{p}\right)\frac{n}{3} & n \text{ has prime divisors congruent to 2 mod 3,} \\ & \text{and } p \text{ is the smallest such divisor;} \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

Lemma (I)**.** $ax \equiv b \mod (n)$ has a solution if and only if $\gcd(a,n)$ divides b.

Corollary (From Lemma 3.1 (Sam Edwards))**.** For any $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$ (written invariently), with $|G| = n$,
$$s(G) = \begin{cases} (0, \ldots, 0, \frac{n_r}{2}) & n_r \equiv 0 \text{ mod } 2 \text{ and } n_{r-1} \equiv 1 \text{ mod } 2; \\ 0 & \text{otherwise.} \end{cases}$$

New Conjectures, Propositions, and Proofs

Lemma 1. For any set A and positive integer $h \leq |A|$,

$$|h\hat{}A| \geq |A| - h + 1.$$

PROOF. Write $A = \{a_0, a_1, \ldots, a_m\}$. Then observe that

$$\begin{aligned} b_h &= a_0 + \cdots + a_{h-1} + a_h, \\ b_{h+1} &= a_0 + \cdots + a_{h-1} + a_{h+1}, \\ &\vdots \\ b_{m-1} &= a_0 + \cdots + a_{h-1} + a_{m-1}, \\ b_m &= a_0 + \cdots + a_{h-1} + a_m, \end{aligned}$$

are all distinct since $a_h, \ldots, a_m$ are all distinct. Thus,

$$\{b_h, b_{h+1}, \ldots, b_{m-1}, b_m\} \subseteq h\hat{}A$$

so $|h\hat{}A| \geq m - (h-1) = |A| - h + 1$.

□

3/18/19

Proposition 2. For all groups G with order n, and for all positive integers $k > l$,

$$\mu\hat{}(G, \{k, l\}) \leq \left\lfloor \frac{n - 2 + l + k}{2} \right\rfloor.$$

PROOF. Write A for a (k, l)-sum-free subset of G where $|A| = m = \mu\hat{}(G, \{k, l\})$ and $n = |G|$. Using Lemma 1,

$$\begin{aligned} n &\geq |k\hat{}A| + |l\hat{}A| \\ &\geq m - k + 1 + m - l + 1 \\ &\geq 2m - (k + l) + 2 \\ \implies m &\leq \frac{n - 2 + k + l}{2} \\ \implies \mu\hat{}(G, \{k, l\}) &\leq \left\lfloor \frac{n - 2 + k + l}{2} \right\rfloor. \end{aligned}$$

□

3/18/19

Proposition 3. For all $|G| = n > 2$, with $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, (written invariently),

$$\mu\hat{}(G,\{n-1,1\}) = \begin{cases} n-2 & r \geq 2 \text{ and } n_r \equiv 0 \bmod 2 \text{ and } n_{r-1} \equiv 1 \bmod 2, \text{ or} \\ & r = 1 \text{ and } n \equiv 2 \bmod 4; \\ n-1 & \text{otherwise.} \end{cases}$$

PROOF. By Proposition 2,

$$\mu\hat{}(G,\{2,1\}) \leq \left\lfloor \frac{n-2+n-1+1}{2} \right\rfloor = \left\lfloor \frac{2n-2}{2} \right\rfloor = n-1.$$

Let $A = G \setminus \{\xi\}$ for some $\xi \in G$, so $|A| = n-1$, and $(n-1)\hat{}A \cap 1\hat{}A = \emptyset$ is only satisfied if the sum of the elements of A is ξ. Thus, $\mu\hat{}(G,\{n-1,1\}) = n-1$ if only if there exists some $\xi \in G$ such that $s(G) - \xi = \xi$. In other words, there must be some $\xi \in G$ such that

$$s(G) = 2\xi.$$

1 $\underline{s(G) = 0}$. Then $0 = s(G) = 2\xi$ is satisfied with $\xi = 0$, so $\mu\hat{}(G,\{n-1,1\}) = n-1$.

2 $\underline{s(G) \neq 0}$.

- i $\underline{n_r \equiv 0 \bmod 4}$. Then $\frac{n_r}{2} = s(G) = 2\xi$ is satisfied with $\xi = \frac{n_r}{4}$. Thus, $\mu\hat{}(G,\{n-1,1\}) = n-1$.
- ii $\underline{n_r \equiv 2 \bmod 4}$. Since 2 does not divide $\frac{n_r}{2} \equiv 1 \bmod 2$, there is no such $\xi \in G$. For all $A \subseteq G$ such that $|A| = n-2$, we have

$$(n-1)\hat{}A \cap 1\hat{}A = \emptyset \cap A = \emptyset,$$

so $\mu\hat{}(G,\{n-1,1\}) = n-2$.

□

3/18/19 (rw. 3/27/19)

Proposition 4. For all $|G| = n > 3$, with $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$, (written invariently),

$$\mu\hat{}(G,\{n-2,1\}) = n-2.$$

PROOF. By Proposition 2,

$$\mu\hat{}(\mathbb{Z}_n,\{n-2,1\}) \leq \left\lfloor \frac{n-2+n-2+1}{2} \right\rfloor = \left\lfloor n - \frac{3}{2} \right\rfloor = n-2.$$

Let $A = G \setminus \{\xi_1, \xi_2\}$ for some distinct $\xi_1, \xi_2 \in G$, so $|A| = n-2$, and $(n-2)\hat{}A \cap 1\hat{}A = \emptyset$ is only satisfied if the sum of the elements of A is ξ_1, WLOG. Thus, $\mu\hat{}(G,\{n-2,1\}) = n-2$ if only if there exists some distinct $\xi_1, \xi_2 \in G$ such that $s(G) - \xi_1 - \xi_2 = \xi_1$. That is, there must be some distinct $\xi_1, \xi_2 \in G$ such that

$$s(G) = 2\xi_1 + \xi_2.$$

(i) $\underline{n \equiv 1 \bmod 2}$ or $\underline{r \geq 2 \text{ and } n_{r-1} \equiv 0 \bmod 2}$. Then $0 = s(G) = 2\xi_1 + \xi_2$ is satisfied with $\xi_1 = (0,\ldots,0,1)$ and $\xi_2 = (0,\ldots,0,n_r-2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 1$ and $\xi_2 = n-2$) which are distinct since $n_r - 2 \not\equiv 1 \bmod n_r$ for all $n_r > 3$. Thus, $\mu\hat{}(G,n-2,1) = n-2$.

(ii) $\underline{n_r \equiv 0 \bmod 2 \text{ and } n_{r-1} \equiv 1 \bmod 2}$. When $n \neq 6$ then $(0,\ldots,0,\frac{n_r}{2}) = s(G) = 2\xi_1 + \xi_2$ is satisfied with $\xi_1 = (0,\ldots,0,1)$ and $\xi_2 = (0,\ldots,0,\frac{n_r}{2}-2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 1$ and $\xi_2 = \frac{n}{2}-2$) which are distinct since $\frac{n}{2} - 2 \neq 1$ for all $n \neq 6$. When $n = 6$, take $\xi_1 = (0,\ldots,0,5)$ and $\xi_2 = (0,\ldots,0,2)$ (if $G \cong \mathbb{Z}_n$, $\xi_1 = 5$ and $\xi_2 = 2$). Thus, $\mu\hat{}(G,n-2,1) = n-2$.

□

3/18/19 (rv. 3/20/19) (rw.3/28/19)

Proposition 5. For any G with $|G| = n \equiv 0 \bmod 2$,

$$\hat{\mu}(G, \{2, 1\}) = \frac{n}{2}.$$

PROOF. Write $G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}$. By Proposition 2,

$$\hat{\mu}(G, \{2, 1\}) \leq \left\lfloor \frac{n - 2 + 2 + 1}{2} \right\rfloor = \frac{n}{2}.$$

If $n \equiv 0 \bmod 2$, there is some $n_i \equiv 0 \mod 2$, so we can take $A \subseteq G$ to be the set with all the elements of G whose ith element is congruent to 1 mod 2. The ith entry of the sum of any two elements in A will be congruent to 0 mod 2, so $2\hat{}A \cap 1\hat{}A = \emptyset$. Thus,

$$\hat{\mu}(G, \{2, 1\}) \geq |A| = n_1 \cdot \cdots \cdot n_{i-1} \cdot \frac{n_i}{2} \cdot n_{i+1} \cdot \cdots \cdot n_r = \frac{n}{2}.$$

□

3/26/19 (rv. 4/4/19)

NOTE: This means that by Proposition G.21, $\hat{\mu}(\mathbb{Z}_2^r, \{2, 1\}) = 2^{r-1} = \mu(\mathbb{Z}_2^r, \{2, 1\})$.

Conjecture 6. For all positive integers n_1 and n_2,

$$\hat{\mu}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2, 1\}) = \begin{cases} \mu & \text{if } \exists p \in \mathbb{P},\ p|n,\ p \equiv 2 \bmod 3 \\ \mu + 1 & \text{otherwise.} \end{cases}$$

When n_1 does not divide n_2, $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \cong \mathbb{Z}_{n_1 n_2} = \mathbb{Z}_n$, so by Theorem G.67 (Zannier; cf. [205]),

$$\hat{\mu}(\mathbb{Z}_n, \{2, 1\}) = \begin{cases} \left(1 + \frac{1}{p}\right) \frac{n}{3} \\ \\ \left\lfloor \frac{n}{3} \right\rfloor + 1 \end{cases} = \begin{cases} v_1(n, 3) \cdot \frac{n}{n} \\ \\ v_1(n, 3) \cdot \frac{n}{n} + 1 \end{cases} \overset{\text{G.18}}{=} \begin{cases} \mu(\mathbb{Z}_n, \{2, 1\}) & \text{if } \exists x \in \mathbb{P},\ x|n,\ x \equiv 2 \bmod 3, \\ & \text{and } p \text{ is the smallest such } x \\ \mu(\mathbb{Z}_n, \{2, 1\}) + 1 & \text{otherwise.} \end{cases}$$

When $n \equiv 0 \bmod 2$, clearly the smallest prime divisor of n congruent to 2 mod 3 is 2, so by Proposition 5 and Proposition G.18,

$$\hat{\mu}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2, 1\}) \overset{5}{=} \frac{n}{2} = \left(1 + \frac{1}{2}\right) \frac{n}{3} = v_1(n, 3) \cdot \frac{n}{n} \overset{\text{G.18}}{=} \mu(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2, 1\}).$$

Now we should consider when $n \equiv 1 \bmod 2$.

Proposition 7. For any positive integer $w \equiv 1 \bmod 2$,

$$\mu\hat{}(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) \geq 3w+1.$$

PROOF. Consider the sets

$$\begin{aligned} A_0 &= \{0\} \times \{-w, -w+2, \ldots, w-2, w\}, \\ A_1 &= \{1\} \times \{0, 2, \ldots, 2w-4, 2w-2\}\,, \text{ and} \\ A_2 &= \{2\} \times \{-2w+2, -2w+4, \ldots, -2, 0\}\,, \end{aligned}$$

and let $A = A_0 \cup A_1 \cup A_2$. Observe that A_0, A_1, and A_2 are disjoint, so

$$|A| = |A_0| + |A_1| + |A_2| = \left(\frac{w-(-w)}{2} + 1\right) + (w-1-0+1) + (w-1-0+1) = w+1+w+w = 3w+1.$$

We can recognize the elements in A_0, A_1, and A_2 as arithmetic sequences (with a common difference of 2), so we can easily write

$$\begin{aligned} 2\hat{}A_0 &= \{0\} \times \{-2w+2, -2w+4, \ldots, 2w-4, 2w-2\}, \\ A_1 + A_2 &= \{0\} \times \{-2w+2, -2w+4, \ldots, 2w-4, 2w-2\}, \\ 2\hat{}A_2 &= \{1\} \times \{-4w+6, -4w+8, \ldots, -4, -2\}, \\ A_0 + A_1 &= \{1\} \times \{-w, -w+2, \ldots, 3w-4, 3w-2\}, \\ 2\hat{}A_1 &= \{2\} \times \{2, 4, \ldots, 4w-8, 4w-6\}, \text{ and} \\ A_0 + A_2 &= \{2\} \times \{-3w+2, -3w+4, \ldots, w-2, w\}. \end{aligned}$$

Notice that since $-4w \equiv -w \bmod 3w$ and $-3w \equiv 0 \bmod 3w$, $2\hat{}A_0 = A_1 + A_2$, $2\hat{}A_2 \subset A_0 + A_1$, and $2\hat{}A_1 \subset A_0 + A_2$. Now we only must show that

$$A_0 \cap A_1 + A_2 = \emptyset,\ A_1 \cap A_0 + A_1 = \emptyset, \text{ and } A_2 \cap A_0 + A_2 = \emptyset.$$

In $\mathbb{Z}_{3w}$, $-2w \equiv w$, so we can recognize that the elements of $A_1 + A_2$ continue the arithmetic sequence in A_0 and since $2w \equiv -w$, the elements of A_0 continue the arithmetic sequence in $A_1 + A_2$. The same is true for $A_0 + A_1$ with A_1, and $A_0 + A_2$ with A_2. The three sequences are the same, since they all contain 0 and have a common difference of 2, and repeat in $3w$ terms (because $3w \equiv 1 \bmod 2$). Because the sequence has $3w$ unique terms,

$$A_0 \cap A_1 + A_2 = \emptyset,\ A_1 \cap A_0 + A_1 = \emptyset, \text{ and } A_2 \cap A_0 + A_2 = \emptyset.$$

□

4/9/19

NOTE: By Theorem G.18, if w has no prime divisor congruent to 2 mod 3,

$$\mu\hat{}(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \{2,1\}) = 3w+1 = \left\lfloor \frac{3w}{3} \right\rfloor \cdot 3 + 1 = v_1(3w,3) \cdot \frac{9w}{3w} \overset{\text{G.18}}{=} \mu(\mathbb{Z}_3 \times \mathbb{Z}_{3w}, \{2,1\}) + 1.$$

Proposition 8. For all positive $\kappa \equiv 1 \bmod 6$,

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) \geq \frac{\kappa-1}{3} \cdot \kappa + 1.$$

PROOF. Write

$$B = \left\{1 - \frac{\kappa-1}{3}, 3 - \frac{\kappa-1}{3}, \ldots, \frac{\kappa-1}{3} - 3, \frac{\kappa-1}{3} - 1\right\}$$

and consider the sets

$$\begin{aligned}
A_0 &= \{0\} \times \left(B \cup \left\{\frac{\kappa-1}{3} + 1\right\}\right), \\
A_1 &= \{1\} \times B, \\
A_2 &= \{2\} \times B, \\
&\vdots \\
A_{\kappa-2} &= \{\kappa-2\} \times B, \text{ and} \\
A_{\kappa-1} &= \{\kappa-1\} \times B,
\end{aligned}$$

and take $A = \bigcup_{i=0}^{\kappa-1} A_i$. We can see that $|A| = \left(\frac{\kappa-1}{3}\right) + 1 + (\kappa-1)\left(\frac{\kappa-1}{3}\right) = \kappa\left(\frac{\kappa-1}{3}\right) + 1$. We will show that A is weak $(2,1)$-sum-free. Notice that elements of B form an arithmetic sequence with a common difference of 2, so any two elements of $A^* = A \setminus \{(0, \frac{\kappa-1}{3})\}$ will sum to an element whose second coordinate is in

$$\begin{aligned}
C &= \left\{2 - \frac{2\kappa-2}{3}, 4 - \frac{2\kappa-2}{3}, \ldots, \frac{2\kappa-2}{3} - 4, \frac{2\kappa-2}{3} - 2\right\} \\
&= \left\{2 - \frac{2\kappa-2}{3}, 2 - \frac{2\kappa-2}{3} + (2), \ldots, 2 - \frac{2\kappa-2}{3} + \left(\frac{4\kappa-4}{3} - 6\right), 2 - \frac{2\kappa-2}{3} + \left(\frac{4\kappa-4}{3} - 4\right)\right\},
\end{aligned}$$

whose elements also form an arithmetic sequence with a common difference of 2. Observe that the sequence in C continues the sequence in B (without the term $\frac{\kappa-1}{3} + 1$) and has

$$\frac{\frac{4\kappa-4}{3} - 4}{2} + 1 = \frac{2\kappa-2}{3} - 1$$

terms, while the sequence in B has $\frac{\kappa-1}{3}$ terms. The full sequence, $(0, 2, \ldots, \kappa-4, \kappa-2)$, repeats in a minimum of κ terms (since $\kappa \equiv 1 \bmod 2$), and because

$$|B| + |C| = \frac{\kappa-1}{3} + \frac{2\kappa-2}{3} - 1 = \frac{3\kappa-3}{3} - 1 = \kappa - 2 < \kappa,$$

we know that $B \cap C = \emptyset$. This shows that $(A^* + A^*) \cap A = \emptyset$. Now we just must show that

$$\left(A^* + \left\{\left(0, \frac{\kappa-1}{3} + 1\right)\right\}\right) \cap A = \emptyset,$$

or equivalently, that for all $i \in \{0, 1, \ldots, \kappa-2, \kappa-1\}$, for all $x \in (A_i \cap A^*)$,

$$x + \left(0, \frac{\kappa-1}{3} + 1\right) \notin A_i.$$

Observe that for all such i, for all $x \in (A_i \cap A^*)$,

$$x + \left(0, \frac{\kappa-1}{3} + 1\right) \in (\{i\} \times B) + \left\{\left(0, \frac{\kappa-1}{3} + 1\right)\right\} = \{i\} \times \left\{2, 4, \ldots, -3 - \frac{\kappa-1}{3}, -1 - \frac{\kappa-1}{3}\right\} = \{i\} \times D.$$

The elements of D also form an arithmetic sequence with a common difference of 2 and the sequence in B continues the sequence in D. Again, the full sequence, $(0, 2, \ldots, \kappa-4, \kappa-2)$, repeats in a minimum of κ terms (since $\kappa \equiv 1 \bmod 2$), and because

$$|B| + |D| = \frac{\kappa-1}{3} + \frac{\kappa-1}{3} = \frac{2\kappa-2}{3} < \kappa,$$

we know that $B \cap D = \emptyset$. Considering $i = 0$, we must show that $\left\{\frac{\kappa-1}{3} + 1\right\} \cap D = \emptyset$: recognize that $-1 - \frac{\kappa-1}{3} \equiv 2\left(\frac{\kappa-1}{3}\right) \bmod \kappa$ and since $\kappa \equiv 1 \bmod 6$, $\frac{\kappa-1}{3} \equiv 0 \bmod 2$. This means that

$$2\left(\frac{\kappa-1}{3}\right) - \frac{\kappa-1}{3} = \frac{\kappa-1}{3} \in D.$$

Since $|D| = \frac{\kappa-1}{3} < \kappa$, $\frac{\kappa-1}{3} + 1 \notin D$, so we are done.

$\square$

4/23/19

NOTE: For all κ with no prime divisors congruent to 2 mod 3,

$$\mu\hat{}(\mathbb{Z}_\kappa^2, \{2,1\}) = \kappa\left(\frac{\kappa-1}{3}\right) + 1 = v_1(\kappa, 3) \cdot \frac{\kappa^2}{\kappa} + 1 \overset{\text{G.18}}{=} \mu(\mathbb{Z}_\kappa^2, \{2,1\}) + 1.$$