EXERCISES ON PERFECTOID SPACES – DAY 2
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4. Adic spaces

(1) (repeated from day 1 because we hadn't defined bounded yet) Let A be a Huber ring.
 (i) If $\Sigma, \Sigma' \subset A$ are bounded subsets, prove that the subset $\Sigma \cdot \Sigma'$ of finite sums of products ss' for $s \in \Sigma$ and $s' \in \Sigma'$ is bounded.

 (ii) Prove that any open subring of A (equipped with the subspace topology) is a Huber ring.

 (iii) Prove that if A_0 is a ring of definition and $a \in A$ is power-bounded then $A_0[a]$ is bounded. Deduce that A^0 is the union of all rings of definition for A.

 (iv) Let B' be an open subring of A and $B \subset A$ a bounded subring that is contained in B'. Construct a ring of definition A_0 satisfying $B \subset A_0 \subset B'$.

(2) We saw via the sup-norm and [BGR, 6.2.4/1] that if A is a reduced k-affinoid algebra then A^0 is bounded. Can you find an example of a reduced Huber ring A such that A^0 is not bounded inside A?

(3) Let A be a k-affinoid algebra.
 (i) For any $v \in X := \mathrm{Spa}(A, A^0)$ corresponding to a valuation on A with rank-1 value group Γ, prove that there is a *unique* continuous multiplicative semi-norm $|\cdot|_v : A \to \mathbf{R}_{\geq 0}$ bounded on A^0 that extends the given absolute value on k.

 (ii) If you are familiar with Berkovich spaces, prove that $v \mapsto |\cdot|_v$ is a continuous surjective map $X \to M(A)$. (This is non-trivial because $\{v \in X \mid v(f) < v(g)\}$ for $f, g \in A$ generally is *not* open in X, even for $A = k\langle t\rangle$. If you get totally stuck, see the proof of 11.1.2 in the notes.)

(4) For any $X = \mathrm{Spa}(A, A^+)$, we recorded that $X(T/s) = \mathrm{Spa}(A(T/s), A^+[T/s])$ for any $s \in A$ and $T = \{f_1, \ldots, f_n\} \subset A$ such that $\sum f_i A$ is open. Likewise, if $(C, C^+) \to (A, A^+)$ and $(C, C^+) \to (B, B^+)$ are maps of *Tate* pairs then the image of $A^+ \otimes_{C^+} B^+$ in $A \otimes_C B$ is an open subring whose integral closure $(A^+ \otimes_{C^+} B^+)^{\sim}$ defines a Tate pair that gives a fiber product (when sheafiness holds).

In general, there is *no reason* to expect that $A^0[T/s] = A(T/s)^0$ nor that the image of $A^0 \otimes_{C^0} B^0 \to A \otimes_C B$ coincides with $(A \otimes_C B)^0$, nor for the analogues with completions. In this exericse, we explore why in the classical rigid case the choice $A^+ = A^0$ *does* interact well with such operations (after completion)!

(i) Using the non-trivial result from [BGR, §6.3] that a map $R \to S$ between k-affinoids is integral (e.g., surjective or an isomorphism) if and only if $R^0 \to S^0$ is integral, prove that the integral closure of the image of $A^0 \widehat{\otimes}_{C^0} B^0 \to A \widehat{\otimes}_C B$ is $(A \widehat{\otimes}_C B)^0$. Hint: reduce to the case $C = k$.

(ii) If $T = \{f_1, \dots, f_n\} \subset A$ generates the unit ideal then prove $A^0\langle T/g\rangle \to A\langle T/g\rangle$ has image with integral closure $A\langle T/g\rangle^0$. (Recall $A^0\langle T/g\rangle$ means the ϖ-adic completion of $A[T/g] \subset A[1/g]$ for any pseudo-uniformizer ϖ of k.) It really can happen that $A^0\langle T/g\rangle \hookrightarrow A\langle T/g\rangle^0$ is not an equality; see 12.2.8 in the notes.

(5) For c satisfying $0 < |c| < 1$, consider the type-2 point $v = v_{c,|c|} \in \mathbf{D}_k$. Identify its residue field with $\kappa(\tau)$ where τ is the reduction of $(t-c)/c \in v^{-1}(1)$. Using the identifying of the non-generic points of $\mathrm{RZ}(\kappa(\tau)/\kappa) = \mathbf{P}^1_\kappa$ with rank-2 specializations of v in $\mathbf{D}_k$ (recall $|c| < 1$), let v_∞ be the specialization associated to $\infty \in \mathbf{P}^1_\kappa$.

Check that $Y := \{w \in \mathbf{D}_k \,|\, w(t) \le w(c)\}$ contains v but not the point v_∞ in its closure, so Y is not closed. In particular, its complement $\{w \,|\, w(c) < w(t)\}$ is *not open*. This is a striking contrast with how open neighborhoods of the type-2 point v are defined in $M(k\langle t\rangle)$! Why does this *not imply* that the map $\mathbf{D}_k \to M(k\langle t\rangle)$ is discontinuous at v?

(6) This exercise works out the key input needed to define a fully faithful functor from rigid-analytic spaces over k to adic spaces over $\mathrm{Spa}(k, k^0)$ (structure sheaves will come along for the ride; the real effort is in the topological issues below). For an affinoid rigid-analytic space $X = \mathrm{Sp}(A)$, define $r_k(X) = \mathrm{Spa}(A, A^0)$. If $U \subset X$ is a rational domain then we recorded that $r_k(U) \to r_k(X)$ is an open embedding compatible with the notion of "rational domain" inside $r_k(U)$ and inside its open image in $r_k(X)$.

(i) Prove that if $\{U_1, \dots, U_n\}$ are rational domains covering X then $\{r_k(U_i)\}$ covers $r_k(X)$, and that $r_k(U) \cap r_k(V) = r_k(U \cap V)$ for rational $U, V \subset X$. (Hint: refine to a "rational covering" as in [BGR, 8.2.2/2] for the first assertion.)

(ii) Let $U \subset X$ be *any* affinoid subdomain. Using (i), prove that $r_k(U) \to r_k(X)$ is an open embedding, and that $r_k(U) \cap r_k(V) = r_k(U \cap V)$ for any second affinoid subdomain $V \subset X$. (Hint: use Gerritzen–Grauert)

(iii) Use (ii) and gluing to define r_k from separated rigid-analytic spaces over k to sober topological spaces, and check that (ii) holds with "affinoid subdomain" relaxed to "admissible open subspace". Prove that if $\{U_i\}$ is a collection of admissible open subspaces of X then $\{r_k(U_i)\}$ covers $r_k(X)$ if and only if $\{U_i\}$ is an *admissible cover* of X! (The implication "$\Rightarrow$" is the interesting direction, making Tate's discovery of admissibility all the more amazing.)

(7) Let $X = \mathrm{Spa}(A, A)$ where $A = \mathbf{Z}_p[\![T]\!]$ is given its max-adic (i.e., (p,T)-adic) topology. The continuous map $X \to \mathrm{Spec}(A)$ has exactly one point s over the closed point, corresponding to the trivial valuation on the residue field $A/\mathfrak{m}_A = \mathbf{F}_p$. Note that $X - \{s\}$ has exactly one characteristic-p point, corresponding to $\mathrm{Spa}(\mathbf{F}_p(\!(T)\!), \mathbf{F}_p[\![T]\!])$.

Cousins of the mixed-characteristic (adic) space $X - \{s\}$ play an important role in Scholze's approach to integral p-adic Hodge theory.

(i) For a discrete valuation ring R equipped with its natural topology, show that $\mathrm{Spa}(R, R) \to \mathrm{Spec}(R)$ is a homeomorphism. We visualize the two subsets $\mathrm{Spa}(\mathbf{F}_p[\![T]\!], \mathbf{F}_p[\![T]\!]) = \{p = 0\} \subset X$ and $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p) = \{T = 0\} \subset X$ as vertical and horizontal "axes" with X occupying the first quadrant; show these subsets meet $X - \{s\}$ in $\mathrm{Sp}(\mathbf{F}_p(\!(T)\!))$ and $\mathrm{Sp}(\mathbf{Q}_p)$ respectively.

(ii) Prove that $\{p \neq 0\}$ is covered by the rational domains (visualized as "sectors" in the quadratic X) $Y_n^+ := \{v \in X \mid v(T^n) \leq v(p) \neq 0\}$, and likewise $\{T \neq 0\}$ is covered by the rational domains $Y_n^- := \{v \in X \mid v(p^n) \leq v(T) \neq 0\}$. (Hint: use that p and T are topologically nilpotent in A.) Deduce that $\{p \neq 0\}$, $\{T \neq 0\}$, and $X - \{s\}$ are not quasi-compact.

(iii) Although A is not Tate, show that $(\mathcal{O}_A(Y_n^+), \mathcal{O}_A^+(Y_n^+)) = (B_n[1/p], B_n)$ where B_n is the p-adic completion of $A[T^n/p]$ equipped with its p-adic topology (so $B_n[1/p]$ is Tate!). How about for Y_n^-?

5. Perfectoid fields and rings

(1) Define a sequence of field extensions $\mathbb{Q}_p = K_0 \to K_1 \to \cdots$ by taking $K_{n+1} = K_n(\alpha_n^{1/p})$ where α_n is a uniformizer of K_n. Prove that the completion of $\bigcup_{n=1}^{\infty} K_n$ is a perfectoid field whose tilt is isomorphic to the completed perfect closure of $\mathbf{F}_p((T))$; this gives uncountably many untilts of the latter.

(2) Read the original 4-page paper of Fontaine-Wintenberger on the field of norms (Le "corps des normes" de certaines extensions algébriques de corps locaux, *Comptes Rendus*, 1979; not to be confused with their other article in the same volume!), taking note especially of Lemma 3.1. Then show that if K is a local field of characteristic 0 and L is an algebraic extension of K which is *strictly* APF, then $\widehat{L}$ is perfectoid. We do not know if this continues to be true if L is only assumed to be APF. (By Sen's theorem, any p-adic Lie extension which is not totally unramified is strictly APF.)

(3) Let K be a perfectoid field. Prove that if $K^\flat$ is algebraically closed, then so is K. (Hint: it suffices to check that any monic polynomial $P(T)$ over K° has a root. To do this, use Hensel's lemma to factor $P(T)$ so as to separate out the roots of smallest absolute value, find $\overline{x} \in K^\flat$ such that $P(T - \overline{x}^\sharp)$ has a root of even smaller absolute value, and so on. To see that this converges, you will need to control how much improvement you get at each step.)

(4) Recall that for every perfect $\mathbf{F}_p$-algebra R, there exists a $\mathbb{Z}_p$-flat, p-adically complete and separated ring $W(R)$ such that $W(R)/(p) \cong R$. Show that the reduction map $W(R) \to R$ admits a unique multiplicative section $\overline{x} \mapsto [\overline{x}]$, which can be computed using the congruence

$$[\overline{x}^{p^n}] \equiv x^{p^n} \pmod{p^{n+1}}$$

where $x \in W(R)$ is any lift of $\overline{x}$. The element $[\overline{x}]$ is called the *Teichmüller lift* of $\overline{x}$.

(5) This exercise gives a crucial extension of Witt vector functoriality used to construct the map θ_A defined in lecture. Let R be a perfect $\mathbf{F}_p$-algebra. Let S be a p-adically separated and complete ring. Let $\pi : S \to S/(p)$ be the canonical projection. Let $t : R \to S$ be a multiplicative map such that $\pi \circ t : R \to S/(p)$ is a ring homomorphism. Prove that the formula

$$T\left(\sum_{n=0}^{\infty} p^n [\overline{x}_n]\right) = \sum_{n=0}^{\infty} p^n t(\overline{x}_n)$$

defines a ring homomorphism $T : W(R) \to S$, which is surjective if $\pi \circ t$ is. (Hint: check additivity modulo p^n by induction on n.)

(6) Let A be a perfectoid ring.
   (i) Show that there is a unique surjective ring homomorphism $\theta_A : W(A^{\flat\circ}) \to A^\circ$ lifting the homomorphism $A^{\flat\circ} \to A^\circ/(p)$.

   (ii) Prove that for $\overline{x} \in A^{\flat\circ}$, $\theta_A([\overline{x}]) = \overline{x}^\sharp$.

(7) For A a perfectoid ring of characteristic p, an element $\xi = \sum_{n=0}^{\infty} p^n [\overline{\xi}_n]$ of $W(A^\circ)$ is *primitive of degree* 1 if $\overline{\xi}_0$ is topologically nilpotent (but not necessarily a unit in A) and $\overline{\xi}_1 \in A^{\circ\times}$. Throughout this exercise, assume that this is the case.
   (i) Show that $\xi_1 := (\xi - [\overline{\xi}_0])/p$ is a unit in $W(A^\circ)$.

   (ii) Show that there exists a unit $u \in W(A^\circ)$ such that $u\xi = p + [\varpi]\alpha$ for some pseudo-uniformizer ϖ of A and some $\alpha \in W(A^\circ)$. That is, ξ is primitive of degree 1 if and only if it generates an ideal which is primitive of degree 1 according to the definition given in the lecture.

(8) Let K be a perfectoid field of characteristic 0.
   (i) Prove that $\ker(\theta)$ contains an element ξ which is primitive of degree 1. (Hint: choose $\overline{\xi}_0 \in K^{\flat\circ}$ such that $\overline{\xi}_0^\sharp/p \in K^{\circ\times}$, then take $\xi = [\overline{\xi}_0] + p\xi_1$ where $\theta(\xi_1) = -\overline{\xi}_0^\sharp/p$.)

   (ii) Prove that any ξ as in (i) is a generator of the ideal $\ker(\theta)$.

(9) Let F be a perfectoid field of characteristic p and suppose that $\xi \in W(F^\circ)$ is primitive of degree 1.
   (i) Prove that every nonzero element of $W(F^\circ)/(\xi)$ lifts to some element of $W(F^\circ)$ of the form $p^m x$ for some nonnegative integer m and some $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n]$ such that $\overline{\xi}_0$ does not divide $\overline{x}_0$ in F°.

   (ii) For x as in (i), put $\xi_1 := (\xi - [\overline{\xi}_0])/p$ and $x_1 := (x - [\overline{x}_0])/p$. Show that $x - \xi_1^{-1}\xi x_1$ equals $[\overline{x}_0]$ times a unit in $W(F^\circ)$.

   (iii) Deduce that ξ generates a prime ideal of $W(F^\circ)$ and a maximal ideal of $W(F^\circ)[p^{-1}]$.

   (iv) Put $K := W(F^\circ)[p^{-1}]/(\xi)$. Show that K is a perfectoid field with $K^\flat \cong F$.

(10) Let K be a perfectoid field. Let F be a finite extension of $K^\flat$ and define the ring

$$L := W(F^\circ) \otimes_{W(K^{\flat\circ}),\theta} K.$$

(i) Suppose that F is Galois over $K^\flat$ with group G. Show that the invariant subring L^G equals K. (Hint: there is only an issue in the characteristic 0 case, where we may average over orbits of G.)

(ii) Show that L is a finite extension of K of degree $[F : K^\flat]$. (Hint: a lemma of Artin asserts that any field equipped with an action of a finite group is Galois over the fixed subfield.)

(11) Let K be a perfectoid field.

(ii) Show that every finite extension of K is perfectoid.

(iii) Deduce the *tilting equivalence*: the functor $L \mapsto L^\flat$ defines an equivalence of categories between finite étale K-algebras and finite étale $K^\flat$-algebras, and hence an isomorphism of absolute Galois groups $G_K \cong G_{K^\flat}$.

6. Almost Ring Theory

(1) Review the equivalences between the following definitions for a ring map $R \to S$ to be formally unramified:

(a) Given a commutative square

$$\begin{array}{ccc} S & \longrightarrow & B/I \\ \uparrow & & \uparrow \\ R & \longrightarrow & B \end{array}$$

where B is a ring and $I \subset B$ is an ideal with $I^2 = 0$, there exists at most one map $S \to B$ making the diagram commute.

(b) $\Omega_{S/R} = 0$.

(c) There exists an element $e \in S \otimes_R S$ such that $e^2 = 1$, $\mu(e) = 1$ and $e \ker(\mu) = 0$. Here $\mu \colon S \otimes_R S \to S$ is the multiplication map.

(2) Assume that p is odd. Let $K = \mathbb{Q}_p(p^{1/p^\infty})$ and $L = K(p^{1/2})$. Show "by hand" that $L^{\circ a}/K^{\circ a}$ is finite étale.

(3) Let $K = \mathbb{C}_p$, and let $\mathfrak{m} = \mathfrak{m}_K$. Is the natural map $K^\circ/p \to \mathrm{Hom}_{K^\circ}(\mathfrak{m}, K^\circ/p)$ an isomorphism? Is it an almost isomorphism?