CH2 EXERCISES

Exercise 1. *Prove that the empty set is a subset of every set*

Proof. For $B \subseteq S \in \mathbf{Set}, B^c \cap B = \emptyset \subseteq S$ ☐

Exercise 2. *Prove the set of all algebraic numbers are countable.*
Hint: For $\sum_0^n a_k z^k = 0, N \in \mathbf{N}$, there are only finitely many equations with

$$n + \sum |a_k| = N$$

Proof. Write $\mathbf{A}$ for the algebraic numbers. Since each algebraic number is determined by its (finite) integral coefficients, then for some n we have inclusions

$$A \hookrightarrow \mathbf{Z}[X] \hookrightarrow \mathbf{Z}^n, a \mapsto a_{n-1}x^{n-1} + ... + a_0 \mapsto (a_0, ..., a_{n-1})$$

Since $\mathbf{Q} \subset \mathbf{A}$, the algebraic numbers are countably infinite. ☐

Exercise 3. *Prove that there exist real numbers which are not algebraic.*

Lemma 1 (Gelfond-Schneider Thm)**.** $\alpha \neq 0, 1, \beta \notin \mathbf{Q}, \alpha^\beta$ *is transcendental.*

Exercise 4. *Is the set of all irrational real numbers countable?*

Proof. Note there is an inclusion $\mathbf{R}/\mathbf{Q} \overset{\iota}{\hookrightarrow} \mathbf{R} \setminus \mathbf{Q}$ of irrational numbers with rational multiple exactly 1. If $\mathbf{R}/\mathbf{Q}$ is countable, $\mathbf{R}$ is a countable union of countable sets $\{xq\}_{q \in \mathbf{Q}}$ and therefore countable, which we know to be untrue. Therefore $\mathbf{R}/\mathbf{Q}$ is uncountable. Since $\iota(\mathbf{R}/\mathbf{Q}) \subset \mathbf{R} \setminus \mathbf{Q}$, the irrational numbers must also be uncountable. ☐

Exercise 5. *Construct a bounded set of real numbers with exactly three limit points.*

Proof. For $n \in \mathbf{N}$,

$$A_0 \overset{\text{def}}{=} \{1/n\}, A_1 \overset{\text{def}}{=} \left\{\frac{n-1}{n}\right\}, A_2 \overset{\text{def}}{=} \left\{\frac{-n+1}{n}\right\}$$

Let A be the union of A_0, A_1, A_2, then $A \subset (-1, 1)$ and its limit points are exactly $\{\pm 1, 0\}$. ☐

Exercise 6. *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that $E, \bar{E}$ have the same limit points. Do E, E' always have the same limit points? For any limit point $\ell \in E'$, we have $\ell : N(\ell) \cap E' \subseteq E'; \neq \emptyset$*

Exercise 7. $A_1, A_2, A_3, \ldots$ *subsets of a metric space.*

(a) *For $B_n \overset{def}{=} \bigcup^n A_i$, show $\bar{B_n} = \bigcup^n \bar{A}_i$*
(b) *For $B \overset{def}{=} \bigcup^{\mathbf{Z}} A_i$, show $\bar{B} \subset \bigcup \bar{A}_i$*

Show this inclusion can be proper.

(a) *Consider $n = 2 : \bar{A}_1 \cup \bar{A}_2$*

Exercise 8. *Are points of each open set $E \subseteq \mathbf{R}^2$ limit points? For closed sets?*
For $x \in E, \exists U_x \subset E$, its limit points would be $\ell : N(\ell) \cap E \setminus \{\ell\} \neq \emptyset$.

Exercise 9. *Let $\overset{\circ}{E}$ denote the interior of E.*

(a) *Prove* $\overset{\circ}{E}$ *open*

(b) *Prove* E *open iff* $\overset{\circ}{E} = E$

(c) *For* G *open,* $G \subseteq E$*, prove* $G \subseteq \overset{\circ}{E}$

(d) *Prove* $(\overset{\circ}{E})^c = \bar{E^c}$

(e) *Do* $E, \bar{E}$ *always have the same interiors?*

(f) *Do* $E, \overset{\circ}{E}$ *always have the same closures?*

(a) *By definition,* A *is open if for each point* $p, \exists N(p) \subseteq A$*, and the interior of* A *is exactly such a set.*

(b)

Exercise 10. *Let* X *be infinite. For* $p, q \in X$*, define*

$$d(p,q) \overset{def}{=} \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Prove this is a metric. What are its open, closed and compact subsets?

Proof. $x = y \iff d(x,y) = 0$ so positive definite; if $x \neq y, y \neq x \implies d(x,y) = d(y,x)$ so it's symmetric. To show the triangle inequality holds, break the right-hand side into cases:

$$d(p,r) + d(r,q) = \begin{cases} 2 & p \neq r \text{ and } r \neq q \\ 1 & p = r \neq q \text{ or } p \neq r = q \\ 0 & p = r = q \end{cases}$$

Then for q, r, p pairwise distinct the sum (RHS) is always greater. Otherwise both sides equal 1 (or 0 if all terms equal).

Neighborhoods can't have zero radius, therefore $N(p) = \{q : d(p,q) = 0\} = \{p\}, N(p) \cap A \subseteq \{p\}$ so no limit points, i.e., $A = \bar{A}$. Since $p \in A \implies N(p) \subseteq A, A = \overset{\circ}{A}$, and all subsets are clopen.

Let $\mathcal{U}$ be a cover of A. If A is a finite subset with n elements, we can finitely cover A with $\{U_k \in \mathcal{U} : x_k \in U_k\}_{j:1 \leq j \leq n}$, i.e., (finite) open sets each containing a point of A. Otherwise, for uncountable A, consider the cover $\mathcal{U}' \overset{\text{def}}{=} \{U_j = x_j\}$. Any finite subcovering contains only finite points of A.

Thus, finite sets are compact and infinite sets noncompact. □

Exercise 11. *For* $x, y \in \mathbf{R}$*, define*

- $d_1 \overset{def}{=} (x-y)^2$
- $d_2 \overset{def}{=} \sqrt{|x-y|}$
- $d_3 \overset{def}{=} |x^2 - y^2|$
- $d_4 \overset{def}{=} |x - 2y|$
- $d_5 \overset{def}{=} \frac{|x-y|}{1+|x-y|}$

Which are metrics? Immediately, d_4 *isn't symmetric [*$d(1,4) = 7 \neq 2 = d(4,1)$*] and* d_3 *isn't positive definite [let* $y = -x$*]. Then of the remaining three we check whether or not we have the triangle inequality* $d(x,z) + d(z,y) - d(x,y) \geq 0$*:*

- $(x-z)^2 + (z-y)^2 - (x-y)^2 = x^2 + y^2 + 2z^2 - 2z(x+y) - x^2 - y^2 - 2xy = z^2 + (z-(x+y))^2 + (x+y) - 2xy$

- $$\sqrt{|x-z|}+\sqrt{|z-y|}=\frac{|x-z|-|z-y|}{\sqrt{|x-z|}-\sqrt{|z-y|}}$$
- $$\frac{|x-z|}{1+|x-z|}+\frac{|z-y|}{1+|z-y|}$$

Exercise 12 (Exercise 12)**.** *Prove* $K \overset{def}{=} \{0\} \cup \{1/n\}_{n\in\mathbf{N}}$ *is compact directly from definition.*

Proof. Let U be a open cover of K. For each $x_n \in K, \exists U_n \in U : x_n \in U_n$. Define $U_0 : 0 \in U_0$ to be the open set containing 0. For each $\delta > 0, \exists m \in \mathbf{N} : \{\frac{1}{m}, \frac{1}{m+1}, ...\} \subset U_0 \cap N_\delta$. Thus, there are finitely many $U_n : n < m$ which, in addition to U_0, cover K. □

Exercise 13. *Construct a compact set of reals whose limit points are countable. Something like Cantor set?*

$$f(x) \overset{def}{=} \begin{cases} 1/r & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

Exercise 14 (14)**.** *Give an example of an open cover of* $(0,1)$ *without finite subcover.*

Proof. Let $U \overset{\text{def}}{=} \{(\frac{1}{n}, 1)\}_{n\in\mathbf{N}}$. Then for any $x \in (0,1), \exists M \in \mathbf{N} : x < 1 \implies \frac{1}{M} < x$. So U is a cover of $(0,1)$. However for any finite subset $U' \overset{\text{def}}{=} \{(\frac{1}{n}, 1)\}_{n:1\le n\le k}$ of U with $k < M$, $\frac{1}{M} \notin U'$. Therefore $(0,1)$ is not compact. □

Exercise 15 (15)**.** *Show that Thm 2.36 and its Corollary become false if compact is replaced with either closed or bounded.*

Exercise 16 (16)**.** $E \overset{def}{=} \{p \in \mathbf{Q} : p^2 \in (2,3)\}$. *Show* E *is closed and bounded in* $\mathbf{Q}$*, but not compact. Is* E *open?*

Proof. $1^2 < p^2 < 2^2 \implies E \subset (1,2)$, so E bounded.

Assume E is not closed, and note $\mathbf{Q} \setminus E = ub(E) \cup lb(E)$. Say E has a limit point $\ell \in ub(E) : (\ell-\delta, \ell+\delta) \cap E \neq \emptyset$. Then for some $\ell' \in ub(E) : \ell > \ell'$, set $\delta = \ell - \ell' > 0$. Then we have $(\ell-\delta, \ell+\delta) = (\ell', 2\ell-\ell')$ and $(\ell', 2\ell-\ell') \cap E \neq \emptyset$, which is impossible since $\ell' > p \in E$. A similar argument follows for $\ell \in lb(E)$, thus E is closed.

The cover $U \overset{\text{def}}{=} \{(0, r-\frac{1}{n}) : n \in \mathbf{N}, r^2 \in (2,3)\}$ has no finite subcover, so E is not compact. □

Exercise 17. *Decimal expansions on the unit interval with only 4's and 7's*

$$E \overset{def}{=} \left\{\sum a_k 10^{-k} : a_k \in \{4,7\}\right\}$$

Is E *countable?dense?compact?perfect?*

Proof. There are really two cases here: series including eventually zero terms like $0.44 = 0.44\bar{0}$, and nonterminating series without zeros after the decimal point.

To determine cardinality, we map into sets of known cardinality. In the case including terminating series, E is in surjection with binary decimals covering the unit interval $\sum a_k 10^{-k} \mapsto \sum b_k 2^{-k}$, otherwise there is a bijection in the latter case. So E is uncountable either way.

E is not dense in the unit interval because there are no terms of E in the interval $(0.5, 0.6)$.

We first show E is perfect. Assume contrary, then there exists $\ell \notin E : N_\epsilon(\ell) \cap E \neq \emptyset$. Then ℓ has a decimal expansion with digits which aren't 4,7, or in certain cases, 0 [for example, 0.404]. If the first such digit occurs at $k = n$, then for any $s \in E, |\ell - s| > 10^{-m-1}$. Therefore ℓ cannot be a limit point, and E is perfect.

Since E is perfect, it contains its limit points and is therefore closed. Either $(0.4, 0.7)$ or $(0.\bar{4}, 0.\bar{7}) = (\frac{4}{9}, \frac{7}{9})$ is a bound for E, and since E is closed and bounded, E is compact. □

Exercise 18 (18). *Is there a nonempty perfect set in* $\mathbf{R}$ *which contains no rational number? Assume there is such a set* P*, then for* $q \in \mathbf{Q}, \epsilon > 0$*, the intersection*

$$N_\epsilon(q) \cap (P \setminus \{q\})$$

must be empty.

Since P *is nonempty, we have* $p \in P : p \in (p - \epsilon, p + \epsilon)$*. We show that any open interval* (a, b) *contains rational points. [WTS open segment has rationals]*

Exercise 19 (19). *Prove:*

(a) Disjoint closed sets A, B *in a metric space* X *are separated*
(b) Prove the same for disjoint open sets.
(c) For some $p \in X, A \overset{def}{=} \{q : d(p, q) < \delta\}, B \overset{def}{=} \{q : d(p, q) > \delta\}$*.* A, B *are separated.*
(d) Every connected metric space with at least two points is uncountable.
(a) For disjoint closed sets, $A \cap B = \emptyset, A = \bar{A}, B = \bar{B}$*. Then* $\bar{A} \cap \bar{B} = \emptyset$
(b) For disjoint open subsets A, B*,* $A \cup B$ *also open*

Exercise 20 (20). *Are closures and interiors of connected sets connected?*

Exercise 21 (21). *Let* $A, B \subset \mathbf{R}^k$ *be separated subsets. Then for* $a \in A, b \in B, t \in \mathbf{R}$*, define*

$$p(t) = (1 - t)a + tb, A_o = p^{-1}(A), B_0 = p^{-1}(B)$$

Prove

(a) A_0, B_0 *are separated subsets.*
(b) $\exists t_0 \in (0, 1) : p(t_o) \notin A \cup B$
(c) Every convex subset of $\mathbf{R}^k$ *is connected.*

Rewrite as $p(t) = a + t(b - a)$*. On the closed unit interval,* $p(t)$ *deforms* a *into* b *and vice-versa.*

Exercise 22 (22). *A metric space is* <u>*separable*</u> *if it contains a countable dense subset. Prove* $\mathbf{R}^k$ *is separable. TBD:Show cart product of intervals interior* $N_\delta(x) \subset \mathbf{R}^k$

For $x = (x_1, ..., x_k) \in \mathbf{R}^k, E \overset{def}{=} \{(q_i, q_j)_{i,j:1 \leq i,j \leq k} \subseteq \mathbf{Q}^k : q_i, q_j \in \mathbf{Q}\}$*, the cartesian product of open intervals, we will show* $N_\delta(x) \cap E$ *is an open subset of* $\mathbf{Q}^k$ *and* $\mathbf{R}^k$ *is thus separable.*
For each $\delta > 0, \exists n \in \mathbf{N} : 1/n < \delta$*. Then* $B_{1/n}(x) \subset B_\delta(x)$*.*

Exercise 23 (23). *Prove that every separable metric space has a countable base.*

Exercise 24 (24)**.** *Let X be a metric space in which every infinite subset has a limit point. Prove X is separable.*

Exercise 25 (25)**.** *Prove every compact metric space K has countable base, and that K is therefore separable.*

Exercise 26 (26)**.** *Let X be a metric space in which every infinite subset has a limit point. Prove X is compact.*

Exercise 27 (27)**.** *For $E \subseteq \mathbf{R}^k$ uncountable and P its condensation points, prove P is perfect and $P^c \cap E$ at most countable.*

Exercise 28 (28)**.** *Prove that every closed set in a separable metric space is the union of a (perhaps empty) perfect set and an at most countable set. (corollary: every countable closed subset of $\mathbf{R}^k$ has isolated points) hint: ex27*

Exercise 29 (29)**.** *Prove that open sets in $\mathbf{R}$ are the union of at most countable intervals. Define a relation on $U \subseteq \mathbf{R}$ as follows: $\forall x, y \in U, x \sim y \iff$ there is a closed interval of U with endpoints x, y. This relation is reflexive and symmetric, and for $x \sim z, z \sim y$, the union of closed segments is again in U.*

Exercise 30 (30)**.** *Imitate the proof of Thm 2.43 to obtain the following result:*
If $\mathbf{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset then at least one F_n has nonempty interior.
Equivalent statement: If G_n is a dense open subset of $\mathbf{R}^n$, then $\bigcap_1^\infty G_n \neq \emptyset$ (in fact, dense in $\mathbf{R}^n$).

DEFINITIONS AND RESULTS

limit point We say ℓ is a <u>limit point</u> of E if there exists $\{q : q \neq \ell\} \subset E \cap N_r(\ell)$

closure The closure $\bar{A}$ of $A \subseteq X$ is A plus its limit points.

compact A subset $S \subseteq X$ is <u>compact</u> if each cover $\{U_i\}_I$ admits a finite subindexing $\{U_i\}_{J \subseteq I}$, i.e., a finite subcover.

perfect A subset is <u>perfect</u> if it is closed and each point is a limit point.

dense A is <u>dense</u> in X if for any other subset $B, B \cap \bar{A} \neq \emptyset$.

condensation point A point p such that $|N_r(p) \cap E| = \mathfrak{c}$

separate A, B are <u>separated</u> if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$

connected S is <u>connected</u> if it is not the disjoint union of separable sets.

base A collection of open subsets $\{V_a\}$ such that every open set is some union $\bigcup V_a$.

We want to show there exists a nontrivial open subset of $\mathbf{Q}^k$ interior to every open subset of $\mathbf{R}^k$.

Assuming Euclidean metric, set $\delta = \frac{\sqrt{n}}{n^2}\epsilon$ and define $\Delta \overset{def}{=} (\delta, ..., \delta) \in \mathbf{R}^k$.

$$d(x, x+\Delta) = \sqrt{\sum \left(x_i - (x_i + \frac{\sqrt{n}}{n^2}\epsilon)\right)^2} = \frac{\epsilon}{n} \in N_\epsilon(x)$$

By the archimidean property of $\mathbf{R}$, there exists $M \in \mathbf{N}, M > 1$:

$$\frac{\sqrt{n}}{n^2}\epsilon < 1 \implies M^{-1} < \frac{\sqrt{n}}{n^2}\epsilon$$

Therefore, we have the open subset $E \subset \mathbf{Q}^k$:

$$E \overset{def}{=} \mathbf{Q}^k \cap \prod_{i=1}^{n}(x_i - \delta, x_i + \delta) \subset \prod_{i=1}^{n}(x_i - \delta, x_i + \delta) \subset N_\epsilon(x)$$

Moreover for $(M^{-1}, ..., M^{-1}) \in N_\epsilon(0)$, so E is nonempty. [show $N_\epsilon(x) \subset U(x)$]

WTS (countable) rationals in any real open set. Any real open set is the disjoint product of at most countable open intervals. Since $\mathbf{R} \cong (0,1)$, it suffices to prove this on the unit interval. Let $(a,b) \subset (0,1)$. Then $0 < \frac{1}{M} < b - a < 1, M \in \mathbf{N}$.

From this we can show $b - M^{-1} > a, a + M^{-1} < b \implies a + M^{-1} - b + M^{-1} < b - a, -(b-a) + 2$ For any $x \in (a,b), x - a, b - x > 0, 2x + b - a > 0$ WTS $\exists \delta > 0 : N_\delta(x) \cap U_x \neq emptyset$

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          1
3 + ---------
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THINGS WHICH ARE ACTUALLY WRONG

Let U, L be the open upper and open lower halves of $\mathbf{R}^k$, i.e., given $x = (x_1, ..., x_n), (0, ..., 0) \in \mathbf{R}^k, \begin{cases} x_1 > 0 \implies & x \in U \\ x_1 < 0 \implies & x \in L \\ x_1 = 0 \implies & x \notin U \cup L \end{cases}$. Then for $\delta > 0, (\pm\delta, 0, ..., 0) \in N_r(0) \cap (A \subset U) N_r(0) \cap (B \subset L), 0 \in \bar{U} \cap \bar{L}$ therefore by definition $(U \cap \bar{L}) \cap (\bar{U} \cap L) = \emptyset$. We will show these definitions are equivalent.