GEOMETRY PROBLEM

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> Given four points A, B, C, D in the plane such that no three of them are collinear, how many angles are determined by these points? If we measured all the angles and recorded the smallest angle measure, how large could this value be? Discuss where this premise leads mathematically and describe how you might base a miniature student research project on this problem.

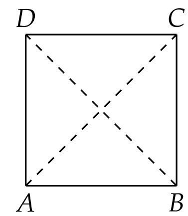

Figure 1: The square is the limiting case which provides the solution. Imagine stretching D or B out to increase the angle at A and C. The angles at D or B would decrease. The square is a "stable" solution at which all of the angles are maximized subject to the constraint that it is the smallest angle.

There are 12 angles. For proof, consider that an ordered set of three distinct points makes an angle. Any set of three points are not collinear by assumption (we will assume that they are distinct as well otherwise they would only vacuously satisfy the assumption). There are ${}_4P_3/2 = 12$ angles since the angle formed by the points (a, b, c) is equivalent to the angle formed by (c, b, a). Since any set of three, of the four points, are not collinear, they form a triangle. The number of triangles is ${}_4C_3 = 4$ which means we are dealing with a quadrilateral. Four of the angles are the angles of the quadrilateral. The other eight are from the triangles formed by making diagonals. Each of these angles is smaller than at least one of the other four since they are each contained within one of the four. The four largest angles are subject to $\sum_{i=1}^{4} m\angle a_i = 360$ where $\forall i \; 0 < m\angle a_i < 180$. For any four tuple which satisfies this property, the supremum of the minimum of the tuples is 90. The concave case is excluded because the smallest angle is less than or equal to the smallest angle of a convex polygon that shares three vertices and encompasses the concave polygon.

Figure 2 is reached by increasing the length of the upper segment, such that $mED = \epsilon$, so that a new polygon $ABCE$ is formed. This is practically equivalent to increasing angle A to $A + \epsilon$ since if $x \approx 0$ then $\sin x \approx x$.[1]

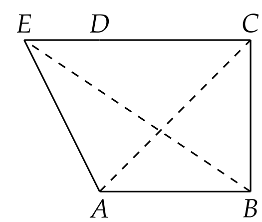

Figure 2: The perturbed square.

[1] This is because the length of ED is equal to a multiple of sine epsilon which is approximately equal to epsilon for small enough values of epsilon as both are close to zero.

Using Figure 3, it is easy to see that

$$m\angle BEC = \tan^{-1}\left(\frac{x}{x+\epsilon}\right) = 45 - \delta(\epsilon)$$

$$m\angle CBE = \tan^{-1}\left(\frac{x+\epsilon}{x}\right) = 45 + \delta(\epsilon)$$

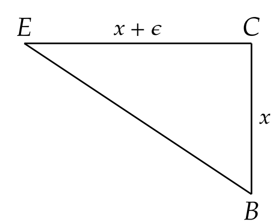

Figure 3: The perturbed 45–45–90 triangle.

since both angles are part of a right triangle. It is also obvious that $m\angle AEB = 45 - \gamma(\epsilon)$. The *pièce de résistance* is that *all* of the angles only depend on the value ϵ. The problem becomes

$$\sup_{\epsilon>0} \min \{45,\, 45 + \delta(\epsilon),\, 45 - \delta(\epsilon),\, 45 - \gamma(\epsilon),\, 45 + \epsilon,\, 90,\, 90 + \epsilon\}$$

at which point it is obvious that the solution is 45°. The minimum will be some value smaller than 45°. The supremum of any of those values will be 45° since as $\epsilon \to 0$ the values all increase as the figure returns to a square.

Possible Research Topics

I think that this problem presents an "in" for research topics on: concave versus convex polygons, quantifiers, and the intricacies of the real number system such as the least upper bound property.

Quantifiers One of the difficulties of the problem is formulating it into precise mathematical language. What are the objects over which I am to take the minimum (angles of a given 4-gon)? What objects am I to maximize over (the set of smallest angles for each quadrilateral in the quadrilaterals)?

Supremum or Maximum This naturally leads to a variety of interesting questions. Why are we assured a minimum angle but may not have a maximum over that set? Why is it natural to take the supremum over the parameter epsilon? What are the possible alternatives and what their relative strengths and weaknesses? Why is it possible to parameterize all of the angles by the amount which the upper segment was perturbed by?

Convexity Does it matter if the polygon is convex or not? This reminds me of Imre Lakatos' *Proofs and Refutation*.

Phase Space and Simulation For a more computationally minded student I might propose simulation to find the solution. Four three tuples could be randomly generated and then the points could iteratively be modified. For example, increase the smallest angle until it is the second smallest or decrease the largest and increase the smallest by equal amounts such that they remain respectively the smallest and largest. The simulation models the movement through the phase space. Does the maximization procedure decide the limiting figures or do they approach the solution? Does the type of distribution used to generate the random points effect the solutions? Is there a procedure which guarantees a solution?