Peter Francis April 19, 2019

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(a)

$$M = \int_V D(\vec{r})\ dV = \int_V Kz\cos^2(\phi)\rho d\rho d\phi dz = K\int_0^{2\pi}\cos^2\phi d\phi\int_0^{2a} zdz\int_0^a \rho d\rho = K\left[\frac{\phi}{2}+\frac{\sin(2\phi)}{4}\right]_0^{2\pi}\left(\frac{(2a)^2}{2}\right)\left(\frac{a^2}{2}\right) = Ka^4\pi$$

$$\implies \boxed{K = \frac{M}{\pi a^4}}.$$

(b)

$$\vec{R} = \frac{1}{M}\int_V \vec{r}D(\vec{r})\ dV = \frac{1}{M}\int_V \vec{r}K\cos^2(\phi)dV = \frac{1}{M}\int_V \vec{r}\frac{M}{\pi a^4}z\cos^2(\phi)dV = \frac{1}{\pi a^4}\int_V \vec{r}z\cos^2(\phi)dV$$
$$= \frac{1}{\pi a^4}\left(\vec{i}\int_V x\cos^2(\phi)zdV + \vec{j}\int_V yz\cos^2(\phi)dV + \vec{k}\int_V z^2\cos^2(\phi)dV\right)$$
$$= \frac{1}{\pi a^4}\left(\vec{i}\int_V \rho z\cos(\phi)\cos^2(\phi)\rho d\rho d\phi dz + \vec{j}\int_V \rho\sin(\phi)z\cos^2(\phi)\rho d\rho d\phi dz + \vec{k}\int_V z^2\cos^2(\phi)\rho d\rho d\phi dz\right)$$
$$= \frac{1}{\pi a^4}\left(\vec{i}\int_0^a \rho^2 d\rho\int_0^{2\pi}\cos^3(\phi)d\phi\int_0^{2a} zdz + \vec{j}\int_0^a \rho^2 d\rho\int_0^{2\pi}\sin(\phi)\cos^2(\phi)d\phi\int_0^{2a} zdz + \vec{k}\int_0^{2a} z^2dz\int_0^{2\pi}\cos^2(\phi)d\phi\int_0^a \rho d\rho\right)$$
$$= \frac{1}{\pi a^4}\left(\vec{i}\left(\frac{a^3}{3}\right)\left[\sin\phi - \frac{\sin^3\phi}{3}\right]_0^{2\pi}\left(\frac{(2a)^2}{2}\right) + \vec{j}\left(\frac{a^3}{3}\right)\left[\frac{-\cos^3\phi}{3}\right]_0^{2\pi}\left(\frac{(2a)^2}{2}\right) + \vec{k}\left(\frac{(2a)^3}{3}\right)\left[\frac{\phi}{2}+\frac{\sin(2\phi)}{4}\right]_0^{2\pi}\left(\frac{a^2}{2}\right)\right)$$
$$= \frac{1}{\pi a^4}\left(\vec{i}\left(\frac{a^3}{3}\right)\cancelto{0}{\left[\sin\phi - \frac{\sin^3\phi}{3}\right]_0^{2\pi}}(2a^2) + \vec{j}\left(\frac{a^3}{3}\right)\left[\frac{-\cos^3\phi}{3}\right]_0^{2\pi}(2a^2) + \vec{k}\left(\frac{8a^3}{3}\right)\left[\frac{\phi}{2}+\frac{\sin(2\phi)}{4}\right]_0^{2\pi}\left(\frac{a^2}{2}\right)\right)$$
$$= \frac{1}{\pi a^4}\left(\cancelto{0}{\vec{j}\frac{2a^5}{3}\left(\frac{-1}{3}+\frac{1}{3}\right)} + \vec{k}\left(\frac{4a^5}{3}\right)\left(\frac{2\pi}{2}\right)\right) = \boxed{\frac{4a}{3}\vec{k}}.$$

For $\vec{R} = X\vec{i} + Y\vec{j} + Z\vec{k}$, $X = 0 = Y$ and $Z = \frac{4a}{3}$.

2

Denote the bottom of the cylinder B, the top T, and the curved wall W. Then,

$$\int_S \vec{A}\cdot d\vec{S} = \int_S \vec{A}\cdot\hat{n}\ dS = \int_B (x\vec{i} - y\vec{j} + z\vec{k})\cdot(-\vec{k})dS + \int_W (x\vec{i} - y\vec{j} + z\vec{k})\cdot\hat{e_\rho}\ dS + \int_T (x\vec{i} - y\vec{j} + z\vec{k})\cdot\vec{k}\ dS$$
$$= -\int_B zdS + \int_W (x\vec{i} - y\vec{j} + z\vec{k})\cdot(\cos\phi\vec{i} + \sin\phi\vec{j})\ dS + \int_T zdS = \cancelto{0}{-\int_B zdS} + \int_W x\cos\phi - y\sin\phi\ dS + \int_T zdS$$
$$= \int_W x\cos\phi - y\sin\phi\ dS + d\int_T dS = \int_W \rho\cos^2\phi - \rho\sin^2\phi\ dS + d\int_T dS$$
$$= \int_0^{2\pi}(\cos^2\phi - \sin^2\phi)d\phi\ \rho\int_0^d\ dz + d\int_0^{2\pi} d\phi\int_0^c d\rho = \left[\frac{\sin(2\phi)}{2}\right]_0^{2\pi}\rho(d) + d(2\pi)(c) = 0 + d(2\pi)(c) = \boxed{2\pi cd}.$$

3

Notice that $\vec{A} = (x^2 + y^2 + z^2)(x\vec{i} + y\vec{j} + z\vec{k}) = r^2\vec{r} = r^2(r\hat{e}_r) = r^3\hat{e}_r$. Then,

(a)

$$\begin{aligned}
\int_S \vec{A}\cdot\ d\vec{S} &= \int_S \vec{A}\cdot\hat{n}\ dS = \int_S r^3\hat{e}_r\cdot\hat{e}_r dS = \int_S r^3 dS = \int_S r^3 r\sin(\theta)d\phi r d\theta = R^5\int_0^\pi \sin(\theta)d\theta\int_0^{2\pi}\ d\phi \\
&= R^5[-\cos(\theta)]_0^\pi(2\pi) = \boxed{4\pi R^5}. \\
\int_S \vec{A}\cdot\ d\vec{S} &= \int_V \nabla\cdot\vec{A}\ dV = \int_V \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 r^3]r^2 dr\sin\theta d\theta d\phi = \int_0^R 5r^4 dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi \\
&= R^5[-\cos(\theta)]_0^\pi(2\pi) = \boxed{4\pi R^5}.
\end{aligned}$$

I have upheld the highest principles of honesty and integrity in all of my academic work and have not witnessed a violation of the Honor Code.