Calculations for *Cosmology ...*

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September 3, 2019

This note shows calculations for the paper *Cosmology from the two-dimensional renormalization group acting as the Ricci flow.* Numerical calculations are performed in the accompanying SageMath notebooks. Sections here that contain numerical results from the notebooks are labeled **SNB**. The section numbering in the notebooks follows the numbering here.

To run the notebooks, either install SageMath from `sagemath.org` or create an account at `cocalc.com` and upload the notebooks to run there. SageMath is a free open-source mathematics software system. Both free and paid accounts are available at `cocalc.com`. Paying for an account supports SageMath (see `reasons-for-purchasing-a-subscription`). Open-source mathematics software such as SageMath is essential for scientific research. Scientific results must be completely open to scrutiny. Closed-source mathematics software blocks scrutiny.

Contents

1 SO(d)-invariant geometry on $I \times S^{d-1}$

We are investigating the fixed-point equation

$$R_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu \tag{1.1}$$

for a Riemannian metric $g_{\mu\nu}$ and a vector field v^μ on $I \times S^3$ both invariant under $\mathbf{SO}(4)$.

1.1 $I \times S^{d-1} \subset \mathbb{R}^d$

Consider $I \times S^{d-1}$ as a spherical shell in $\mathbb{R}^d$ with coordinates x^μ. The euclidean metric is

$$\delta_{\mu\nu} dx^\mu dx^\mu = dr^2 + r^2 ds^2_{S^3} = r^2(d\tau^2 + ds^2_{S^3}) \qquad r = \sqrt{\delta_{\mu\nu} x^\mu x^\mu} \qquad r = e^\tau \tag{1.2}$$

where $ds^2_{S^{d-1}}$ is the $\mathbf{SO}(d)$-invariant metric on the unit $(d-1)$-sphere.

$$\begin{gathered}
\hat{x}^\mu = r^{-1} x^\mu \qquad \partial_\mu r = \hat{x}_\mu \qquad dr = \hat{x}_\mu dx^\mu \\
P_{\mu\nu} = \delta_{\mu\nu} - \hat{x}_\mu \hat{x}_\nu \qquad \partial_\mu \hat{x}_\nu = r^{-1} P_{\mu\nu} \qquad \partial_\mu (r^{-1} \hat{x}_\nu) = r^{-2}(P_{\mu\nu} - \hat{x}_\mu \hat{x}_\nu) \\
\partial_\tau = x^\mu \partial_\mu \qquad d\tau = r^{-1} \hat{x}_\mu dx^\mu \qquad \partial_\mu \tau = r^{-1} \hat{x}_\mu \\
\partial_\alpha = r^{-1} \hat{x}_\alpha \partial_\tau \quad \text{on invariant functions} \\
r^{-2} \delta_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + ds^2_{S^{d-1}} \qquad d\tau^2 = r^{-2} \hat{x}_\mu \hat{x}_\nu dx^\mu dx^\nu \qquad ds^2_{S^{d-1}} = r^{-2} P_{\mu\nu} dx^\mu dx^\nu
\end{gathered} \tag{1.3}$$

1.2 $g_{\mu\nu}$ in conformally flat gauge

The general $\mathbf{SO}(d)$-invariant metric is

$$g_{\mu\nu}(x) dx^\mu dx^\nu = F_1(r)^2 dr^2 + F_2(r)^2 ds^2_{S^{d-1}} \tag{1.4}$$

After a suitable reparametrization $r \to \tau(r)$ the metric is conformally flat

$$\begin{gathered}
g_{\mu\nu} dx^\mu dx^\nu = e^{2f(\tau)} \left(d\tau^2 + ds^2_{S^{d-1}} \right) \\
\frac{dr}{d\tau} = \frac{F_2}{F_1} \qquad F_2(r) = e^{f(\tau)} \\
g_{\mu\nu} = e^{2f(\tau)} r^{-2} \delta_{\mu\nu} = e^{2\tilde{f}(\tau)} \delta_{\mu\nu} \qquad \tilde{f}(\tau) = f(\tau) - \tau
\end{gathered} \tag{1.5}$$

Define

$$f_\tau = \partial_\tau f \tag{1.6}$$

so

$$\partial_\mu \tilde{f} = r^{-1} \hat{x}_\mu \partial_\tau \tilde{f} \qquad \partial_\tau \tilde{f} = f_\tau - 1 \tag{1.7}$$

The covariant derivative is

$$\begin{gathered}
\nabla_\nu w_\mu = \partial_\nu w_\mu - \Gamma^\alpha_{\mu\nu} w_\alpha \\
\Gamma^\alpha_{\mu\nu} = \delta^\alpha_\mu \partial_\nu \tilde{f} + \delta^\alpha_\nu \partial_\mu \tilde{f} - \delta_{\mu\nu} \partial^\alpha \tilde{f} = r^{-1} \left(\delta^\alpha_\mu \hat{x}_\nu + \delta^\alpha_\nu \hat{x}_\mu - \delta_{\mu\nu} \hat{x}^\alpha \right) \partial_\tau \tilde{f}
\end{gathered} \tag{1.8}$$

1.3 $R_{\mu\nu}$

The curvature tensor is

$$R^{\alpha}_{\mu\beta\nu} = \partial_\beta \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\beta}\Gamma^{\alpha}_{\mu\sigma} - (\beta \leftrightarrow \nu) \tag{1.9}$$

The Ricci tensor is

$$\begin{aligned} R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} &= -\delta_{\mu\nu}\partial_\alpha\partial^\alpha \tilde{f} + (d-2)\left(-\partial_\mu\partial_\nu\tilde{f} + \partial_\mu\tilde{f}\partial_\nu\tilde{f} - \delta_{\mu\nu}\partial^\alpha\tilde{f}\partial_\alpha\tilde{f}\right) \\ &= -(d-1)\partial_\tau f_\tau\, r^{-2}\hat{x}_\mu\hat{x}_\nu - \left[\partial_\tau f_\tau + (d-2)\left(f_\tau^2 - 1\right)\right] r^{-2} P_{\mu\nu} \\ R_{\mu\nu}dx^\mu dx^\nu &= -(d-1)\partial_\tau f_\tau\, d\tau^2 - \left[\partial_\tau f_\tau + (d-2)\left(f_\tau^2-1\right)\right] ds^2_{S^{d-1}} \end{aligned} \tag{1.10}$$

The scalar curvature is

$$R = g^{\mu\nu}R_{\mu\nu} = e^{-2f}r^2\delta^{\mu\nu}R_{\mu\nu} = -e^{-2f}(d-1)\left[2\partial_\tau f_\tau + (d-2)\left(f_\tau^2-1\right)\right] \tag{1.11}$$

The Einstein tensor is

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\ G_{\mu\nu}dx^\mu dx^\nu &= \frac{1}{2}(d-2)(d-1)\left(f_\tau^2-1\right)\, d\tau^2 \\ &\quad + \frac{1}{2}(d-2)\left[2\partial_\tau f_\tau + (d-3)\left(f_\tau^2-1\right)\right] ds^2_{S^3} \end{aligned} \tag{1.12}$$

In $d = 4$ dimensions

$$\begin{aligned} R_{\mu\nu}dx^\mu dx^\nu &= -3\partial_\tau f_\tau\, d\tau^2 - \left(\partial_\tau f_\tau + 2f_\tau^2 - 2\right) ds^2_{S^3} \\ R &= -e^{-2f}6\left(\partial_\tau f_\tau + f_\tau^2 - 1\right) \\ G_{\mu\nu}dx^\mu dx^\nu &= 3\left(f_\tau^2-1\right)\, d\tau^2 + \left(2\partial_\tau f_\tau + f_\tau^2 - 1\right) ds^2_{S^3} \end{aligned} \tag{1.13}$$

1.4 $\nabla_\mu v_\nu + \nabla_\nu v_\mu$

The general $\mathbf{SO}(d)$-invariant vector field $v = v^\mu(x)\partial_\mu$ is

$$\begin{aligned} v^\mu\partial_\mu = v^\tau(\tau)\partial_\tau \qquad & v^\mu = v^\tau x^\mu \\ v_\mu dx^\mu = v_\tau d\tau \qquad v_\tau = g_{\tau\tau}v^\tau = e^{2f}v^\tau \qquad & v_\mu = v_\tau r^{-1}\hat{x}_\mu \end{aligned} \tag{1.14}$$

Its covariant derivative is

$$\nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^{\alpha}_{\mu\nu}v_\alpha = (\partial_\tau v_\tau - f_\tau v_\tau)r^{-2}\hat{x}_\mu\hat{x}_\nu + f_\tau v_\tau r^{-2}P_{\mu\nu} \tag{1.15}$$

so

$$\begin{aligned} (\nabla_\mu v_\nu + \nabla_\nu v_\mu)dx^\mu dx^\nu &= (2\partial_\tau v_\tau - 2f_\tau v_\tau)\, d\tau^2 + 2f_\tau v_\tau\, ds^2_{S^{d-1}} \\ 2\nabla_\sigma v^\sigma &= 2a^{-2}\left[\partial_\tau v_\tau + (d-2)f_\tau v_\tau\right] \\ (\nabla_\mu v_\nu + \nabla_\nu v_\mu - \nabla_\sigma v^\sigma g_{\mu\nu})dx^\mu dx^\nu &= (\partial_\tau v_\tau - d f_\tau v_\tau)\, d\tau^2 \\ &\quad - (\partial_\tau v_\tau + (d-4)f_\tau v_\tau)\, ds^2_{S^{d-1}} \end{aligned} \tag{1.16}$$

In $d = 4$ dimensions

$$(\nabla_\mu v_\nu + \nabla_\nu v_\mu - \nabla_\sigma v^\sigma g_{\mu\nu})dx^\mu dx^\nu = (\partial_\tau v_\tau - 4f_\tau v_\tau)\, d\tau^2 - (\partial_\tau v_\tau)\, ds^2_{S^{d-1}} \tag{1.17}$$

2 $R_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ as ode

The fixed point equation $R_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ is equivalent to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu - \nabla_\sigma v^\sigma g_{\mu\nu} \tag{2.1}$$

In the invariant case, using (1.12) and (1.16), this is

$$\begin{aligned} \frac{1}{2}(d-2)(d-1)\left(f_\tau^2 - 1\right) &= \partial_\tau v_\tau - d f_\tau v_\tau \\ (d-2)\partial_\tau f_\tau + \frac{1}{2}(d-2)(d-3)\left(f_\tau^2 - 1\right) &= -\partial_\tau v_\tau - (d-4) f_\tau v_\tau \end{aligned} \tag{2.2}$$

In $d = 4$ dimensions

$$\begin{aligned} 3\left(f_\tau^2 - 1\right) &= \partial_\tau v_\tau - 4 f_\tau v_\tau \\ 2\partial_\tau f_\tau + \left(f_\tau^2 - 1\right) &= -\partial_\tau v_\tau \end{aligned} \tag{2.3}$$

which is the ordinary differential equation

$$\partial_\tau f_\tau = -2 f_\tau v_\tau - 2 f_\tau^2 + 2 \qquad \partial_\tau v_\tau = 4 f_\tau v_\tau + 3 f_\tau^2 - 3 \tag{2.4}$$

3 Analytically continue to real time

Analytically continue from conformal euclidean time τ to conformal real time T

$$\begin{gathered} e^{2f}\left(d\tau^2 + ds^2_{S^{d-1}}\right) = e^{2f}\left(-dT^2 + ds^2_{S^{d-1}}\right) \\ \tau = iT \qquad d\tau = idT \qquad \partial_\tau = i^{-1}\partial_T \\ df = f_\tau d\tau = f_T dT \qquad f_\tau = i^{-1} f_T \qquad v_\tau d\tau = v_T dT \qquad v_\tau = i^{-1} v_T \end{gathered} \tag{3.1}$$

In co-moving time t

$$\begin{gathered} e^{2f}\left(-dT^2 + ds^2_{S^{d-1}}\right) = -dt^2 + a^2 ds^2_{S^{d-1}} \\ dt = e^f dT \qquad a = e^f \qquad \partial_t = a^{-1}\partial_T \qquad \partial_t a = a^{-1}\partial_T a = f_T \end{gathered} \tag{3.2}$$

The Hubble parameter H and the deceleration parameter q are

$$H = a^{-1}\partial_t a = a^{-1} f_T \qquad q = -\frac{a\partial_t^2 a}{(\partial_t a)^2} = -\frac{\partial_T f_T}{f_T^2} \tag{3.3}$$

The ode (2.4) becomes in real time

$$\partial_T f_T = -2 f_T v_T - 2 f_T^2 - 2 \qquad \partial_T v_T = 4 f_T v_T + 3 f_T^2 + 3 \tag{3.4}$$

4 $T_{\mu\nu}$

The fixed point equation $R_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$ is equivalent to Einstein's equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{4.1}$$

with energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{8\pi G} \left(\nabla_\mu v_\nu + \nabla_\nu v_\mu - g_{\mu\nu} \nabla^\alpha v_\alpha \right) \tag{4.2}$$

given in the invariant case by (1.16)

$$T_{\mu\nu} dx^\mu dx^\nu = \frac{1}{8\pi G} \left[\left(\partial_T v_T - 4 f_T v_T \right) dT^2 + \left(\partial_T v_T \right) ds^2_{S^{d-1}} \right] \tag{4.3}$$

This is the energy-momentum tensor of a perfect fluid of density ρ and pressure p

$$\begin{aligned} T_{\mu\nu} dx^\mu dx^\nu &= a^2 (\rho \, dT^2 + p \, ds^2_{S^3}) = \rho \, dt^2 + p \, a^2 ds^2_{S^3} \\ \rho = \frac{1}{8\pi G a^2} \left(\partial_T v_T - 4 f_T v_T \right) & \qquad p = \frac{1}{8\pi G a^2} \partial_T v_T \end{aligned} \tag{4.4}$$

Using the fixed point equation (2.4)

$$\rho = \frac{1}{8\pi G a^2} \left(3 f_T^2 + 3 \right) \qquad p = \frac{1}{8\pi G a^2} \left(4 f_T v_T + 3 f_T^2 + 3 \right) \tag{4.5}$$

The equation-of-state parameter w and the density parameter Ω are

$$w = \frac{p}{\rho} = 1 + \frac{4}{3} \frac{f_T v_T}{f_T^2 + 1} \qquad \Omega = \frac{8\pi G \rho}{3H^2} = 1 + \frac{1}{f_T^2} \tag{4.6}$$

The conservation law $\nabla^\mu T_{\mu\nu} = 0$ in the invariant case is

$$\begin{gathered} T_{\mu\nu} dx^\mu dx^\nu = -a^2 \rho \, d\tau^2 + a^2 p \, ds^2_{S^3} \qquad T_{\mu\nu} = -a^2 \rho \, r^{-2} \hat{x}_\mu \hat{x}_\nu + a^2 p \, r^{-2} P_{\mu\nu} \\ \nabla^\mu T_{\mu\nu} = a^{-2} \partial_\tau (-a^2 \rho) r^{-1} \hat{x}_\nu + (-a^2 \rho) \nabla^\mu (r^{-2} \hat{x}_\mu \hat{x}_\nu) + a^2 p \nabla^\mu (r^{-2} P_{\mu\nu}) \end{gathered} \tag{4.7}$$

$$\begin{aligned} \nabla^\mu (r^{-2} P_{\mu\nu}) &= -a^{-2} (d-1) f_\tau r^{-1} \hat{x}_\nu \\ \nabla^\mu (r^{-2} \delta_{\mu\nu}) &= -2 f_\tau r^{-3} \hat{x}_\nu \\ \nabla^\mu (r^{-2} \hat{x}_\mu \hat{x}_\nu) &= a^{-2} (d-3) f_\tau r^{-1} \hat{x}_\nu \end{aligned} \tag{4.8}$$

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= a^{-2} r^{-1} \hat{x}_\nu \left[\partial_\tau (-a^2 \rho) + (d-3) f_\tau (-a^2 \rho) - (d-1) f_\tau a^2 p \right] \\ &= r^{-1} \hat{x}_\nu \left[-\partial_\tau \rho - (d-1) f_\tau \rho - (d-1) f_\tau p \right] \end{aligned} \tag{4.9}$$

So $\nabla^\mu T_{\mu\nu} = 0$ is

$$\partial_T (a^2 \rho) + (d-3) f_T (a^2 \rho) + (d-1) f_T a^2 p = 0 \tag{4.10}$$

or

$$\partial_T \rho + (d-1) f_T (\rho + p) = 0 \tag{4.11}$$

5 The constant of motion C

If

$$\nabla_\mu v_\nu - \nabla_\nu v_\mu = 0 \tag{5.1}$$

then v^μ is a gradient at least locally

$$v_\mu = \partial_\mu \chi \tag{5.2}$$

In the $\mathbf{SO}(d)$-invariant case v^μ is always a gradient globally, $v_\tau(\tau) = \partial_\tau \chi$.

When v^μ is locally a gradient then

$$\begin{aligned} R_{\mu\nu} &= \nabla_\mu v_\nu + \nabla_\nu v_\mu = 2\nabla_\mu\nabla_\nu\chi \qquad R = 2\nabla_\sigma\nabla^\sigma\chi \\ \nabla^\nu R_{\mu\nu} &= 2\nabla_\mu\nabla^\nu\nabla_\nu\chi + 2R_{\sigma\mu}\nabla^\sigma\chi = \nabla_\mu\left(2\nabla^\nu\nabla_\nu\chi + 2\nabla_\sigma\chi\nabla^\sigma\chi\right) \\ 0 = \nabla^\nu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= \nabla_\mu(\nabla_\sigma\nabla^\sigma\chi + 2\nabla_\sigma\chi\nabla^\sigma\chi) = \nabla_\mu(\nabla_\sigma v^\sigma + 2v_\sigma v^\sigma) \end{aligned} \tag{5.3}$$

so there is a constant of motion

$$C' = \nabla_\sigma v^\sigma + 2v_\sigma v^\sigma = R + 4v_\sigma v^\sigma \qquad \nabla_\mu C' = 0 \tag{5.4}$$

For convenience normalize the constant of motion

$$C = \frac{-2}{(d-1)(d-2)}(\nabla_\sigma v^\sigma + 2v_\sigma v^\sigma) = \frac{-1}{(d-1)(d-2)}(R + 4v_\sigma v^\sigma) \tag{5.5}$$

In the $\mathbf{SO}(d)$-invariant case

$$\begin{aligned} R &= -a^{-2}(d-1)\left[2\partial_\tau f_\tau + (d-2)\left(f_\tau^2 - 1\right)\right] \\ &= -a^{-2}(d-1)\left[-4f_\tau v_\tau - (d-2)\left(f_\tau^2 - 1\right)\right] \\ v_\sigma v^\sigma &= a^{-2}v_\tau^2 \end{aligned} \tag{5.6}$$

so

$$C = a^{-2}\left(-f_\tau^2 - \frac{4}{d-2}f_\tau v_\tau - \frac{4}{(d-1)(d-2)}v_\tau^2 + 1\right) \tag{5.7}$$

Continued to real time

$$C = a^{-2}\left(f_T^2 + \frac{4}{d-2}f_T v_T + \frac{4}{(d-1)(d-2)}v_T^2 + 1\right) \tag{5.8}$$

In $d=4$ dimensions

$$C = a^{-2}\left(f_T^2 + 2f_T v_T + \frac{2}{3}v_T^2 + 1\right) \tag{5.9}$$

6 The $C=0$ cosmological solution

Define

$$h_\pm = f_T + \beta_\pm v_T \qquad \beta_\pm = 1 \pm \frac{1}{\sqrt{3}} \tag{6.1}$$

so that

$$a^2 C = h_+ h_- + 1 \tag{6.2}$$

The ode (3.4) is now

$$\partial_T h_\pm = -1 - h_\pm^2 + \lambda_\pm(h_+h_- + 1) \qquad \lambda_\pm = \frac{\beta_\pm}{\beta_\mp} = 2 \pm \sqrt{3} \tag{6.3}$$

When $C=0$ the ode (6.3) becomes

$$\partial_T h_\pm = -1 - h_\pm^2 \tag{6.4}$$

which has the solution

$$h_- = \cot T \qquad h_+ = -\tan T \tag{6.5}$$

which is unique up to translation and reflection in T. Changing variables back to

$$f_T = \frac{\sqrt{3}}{2}\left(\beta_+ h_- - \beta_- h_+\right) \qquad v_T = \frac{\sqrt{3}}{2}\left(h_+ - h_-\right) \tag{6.6}$$

the $C = 0$ solution becomes

$$\begin{aligned} f_T &= \frac{\sqrt{3}+1}{2}\cot T + \frac{\sqrt{3}-1}{2}\tan T &= \frac{\cos 2T + \sqrt{3}}{\sin 2T} \\ v_T &= -\frac{\sqrt{3}}{2}\cot T - \frac{\sqrt{3}}{2}\tan T &= \frac{-\sqrt{3}}{\sin 2T} \end{aligned} \tag{6.7}$$

Integrating $\partial_T f = f_T$ with constant of integration $\ln t'_0$

$$\begin{gathered} f = \ln t'_0 + (1+\nu)\ln\sin T - \nu\ln\cos T \qquad \nu = \frac{\sqrt{3}-1}{2} \\ a = e^f = t'_0 \sin^{1+\nu} T \cos^{-\nu} T \end{gathered} \tag{6.8}$$

The co-moving time t is

$$\begin{aligned} dt = adT \qquad & \frac{t}{t'_0} = \int_0^T a(T')\,dT' = \frac{1}{2}B_{\sin^2 T}\left(1+\nu/2, 1/2-\nu/2\right) \\ t \in (0, t_{\max}) \qquad & \frac{t_{\max}}{t'_0} = \int_0^{\frac{\pi}{2}} a(T')\,dT' = \frac{1}{2}B(1+\nu/2, 1/2-\nu/2) = 1.470147\ldots \end{aligned} \tag{6.9}$$

where $B(p,q)$ is the Euler beta function and $B_x(p,q)$ is the incomplete beta function

$$\begin{aligned} B_x(p,q) &= \int_0^x s^{p-1}(1-s)^{q-1}ds = 2\int_0^{\sin^2\theta = x} \sin^{2p-1}\theta\ \cos^{2q-1}\theta\ d\theta \\ &= \frac{x^p}{p}\,F(p, 1-q; p+1; x) \\ B(p,q) &= B_1(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \end{aligned} \tag{6.10}$$

F in the second line above is the hypergeometric function ${}_2F_1$.

7 The cosmological parameters [SNB I-1]

7.1 Formulas for H, q, w, and Ω

Substituting in (3.3)

$$\begin{aligned} H = a^{-1}f_T &= \left(t'_0 \sin^{1+\nu} T \cos^{-\nu} T\right)^{-1}\left(\frac{\cos 2T + \sqrt{3}}{\sin 2T}\right) \\ &= \frac{1}{t'_0}\left(\sin^{-2-\nu} T \cos^{-1+\nu} T\right)\frac{1}{2}\left(\cos 2T + \sqrt{3}\right) \\ &= \frac{1}{t'_0}\left(\sin^{-2-\nu} T \cos^{-1+\nu} T\right)\left(\cos^2 T + \nu\right) \end{aligned} \tag{7.1}$$

$$
\begin{aligned}
q = -f_T^{-2}\partial f_T &= -\left(\frac{\cos 2T+\sqrt{3}}{\sin 2T}\right)^{-2}\left(\frac{\sqrt{3}+1}{2}\frac{-1}{\sin^2 T}+\frac{\sqrt{3}-1}{2}\frac{1}{\cos^2 T}\right) \\
&= \frac{2\left(1+\sqrt{3}\cos 2T\right)}{\left(\cos 2T+\sqrt{3}\right)^2} = \frac{\sqrt{3}\cos^2 T-\nu}{(\cos^2 T+\nu)^2}
\end{aligned}
\tag{7.2}
$$

The expansion decelerates ($q>0$) for $T<T_{q=0}$ then accelerates ($q<0$) for $T_{q=0}<T$.

$$
T_{q=0} = \frac{\arccos(-1/\sqrt{3})}{2} = \frac{\pi-\arccos(1/\sqrt{3})}{2} = \frac{\pi-\arctan(\sqrt{2})}{2} = 0.6959\frac{\pi}{2} \tag{7.3}
$$

Substituting in (7.4) gives formulas for w and Ω

$$
\begin{aligned}
w &= 1+\frac{4}{3}\frac{f_T v_T}{f_T^2+1} \\
&= 1+\frac{4}{3}\left(\frac{\cos 2T+\sqrt{3}}{\sin 2T}\right)\left(\frac{-\sqrt{3}}{\sin 2T}\right)\left[1+\left(\frac{\cos 2T+\sqrt{3}}{\sin 2T}\right)^2\right]^{-1} \\
&= \frac{\cos 2T}{3\cos 2T+2\sqrt{3}} \\
\Omega &= 1+\frac{1}{f_T^2} = 1+\left(\frac{\sin 2T}{\cos 2T+\sqrt{3}}\right)^2
\end{aligned}
\tag{7.4}
$$

7.2 Estimate t_0 and t_0'

Use $q_0 = -0.60$ as the present value of the deceleration parameter. Solve $q(T_0) = q_0$ numerically to estimate the present conformal time

$$
T_0 = 0.77\,\frac{\pi}{2} \tag{7.5}
$$

Then calculate the present value of the Hubble parameter by substituting T_0 in (7.1) to fix t_0' in units of the Hubble time $t_H = H_0^{-1}$

$$
t_0' = \left(\sin^{-2-\nu} T_0 \cos^{-1+\nu} T_0\right)\left(\cos^2 T_0+\nu\right) t_H = 1.1\, t_H \tag{7.6}
$$

The Hubble time is

$$
t_H = 4.55\times 10^{17}\text{s} = 1.44\times 10^{10}\text{y} \tag{7.7}
$$

Now t_0, $t_{\max}$, and the present values a_0, w_0, and Ω_0 can be calculated from (6.9), (6.8), and (7.4)

$$
t_0 = 0.73\, t_H \qquad t_{\max} = 1.6\, t_H \qquad a_0 = 1.5\, t_H \qquad w_0 = -0.61 \qquad \Omega_0 = 1.5 \tag{7.8}
$$

Once a_0 is estimated, the redshift can be calculated

$$
z = \frac{a_0}{a} - 1 \tag{7.9}
$$

In particular, $q=0$ at $z=0.18$ and $T=\frac{1}{2}T_0$ at $z=1.69$.

7.3 The limit $t \to 0$

In the limit $T \to 0$

$$f_T \to \frac{1+\nu}{T} \qquad v_T \to \frac{-\sqrt{3}/2}{T}$$
$$f \to (1+\nu)\ln T + \ln t_0' \qquad a \to t_0' T^{1+\nu} \qquad H \to \frac{1+\nu}{t_0'} T^{-2-\nu}$$
$$t \to \frac{t_0'}{2+\nu} T^{2+\nu} \qquad z \to \frac{a_0}{t_0'} T^{-1-\nu} \qquad T \to \left(\frac{2+\nu}{t_0'} t\right)^{1/(2+\nu)} \qquad T \to \left(\frac{t_0'}{a_0} z\right)^{-1/(1+\nu)} \tag{7.10}$$
$$a \to t_0' \left(\frac{2+\nu}{t_0'}\right)^{1/\sqrt{3}} t^{1/\sqrt{3}} \qquad z \to \frac{a_0}{t_0'} \left(\frac{2+\nu}{t_0'}\right)^{-1/\sqrt{3}} t^{-1/\sqrt{3}} \qquad H \to \frac{1+\nu}{2+\nu} t^{-1} \tag{7.11}$$
$$t \to \left(\frac{a_0}{t_0'}\right)^{\sqrt{3}} \left(\frac{2+\nu}{t_0'}\right)^{-1} z^{-\sqrt{3}} \qquad H \to \left(\frac{a_0}{t_0'}\right)^{-\sqrt{3}} \left(\frac{1+\nu}{t_0'}\right) z^{\sqrt{3}}$$

Substituting the estimated values for t_0' and a_0

$$\frac{a}{t_H} \to 1.1\, T^{1.4} \qquad \frac{t}{t_H} \to 0.47\, T^{2.4}$$
$$\frac{a}{t_H} \to 1.7 \left(\frac{t}{t_H}\right)^{.58} \qquad z \to 0.86 \left(\frac{t}{t_H}\right)^{-.58} \tag{7.12}$$
$$\frac{t}{t_H} \to 0.77\, z^{-1.7} \qquad \frac{H}{H_0} \to 0.75\, z^{1.7}$$

7.4 The limit $t \to t_{\max}$

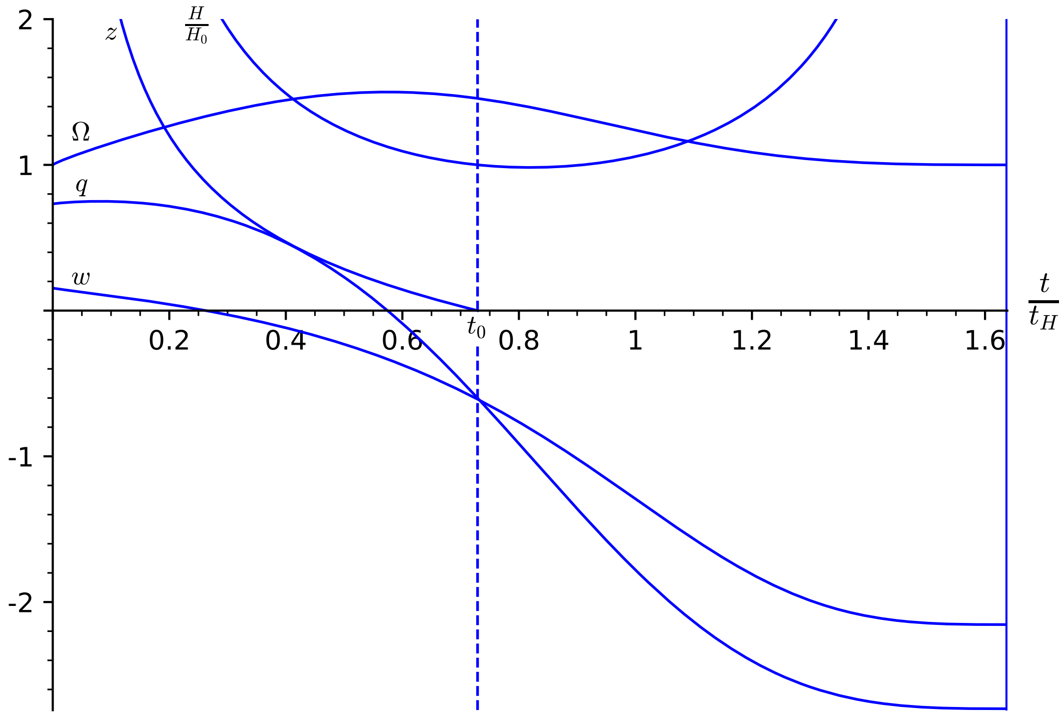
In the limit $T \to \frac{\pi}{2}$

$$f_T \to \frac{\nu}{\frac{\pi}{2} - T} \qquad v_T \to \frac{-\sqrt{3}/2}{\frac{\pi}{2} - T} \qquad f \to -\nu \ln\left(\frac{\pi}{2} - T\right) + \ln t_0''$$
$$a \to t_0'' \left(\frac{\pi}{2} - T\right)^{-\nu} \qquad t \to t_{\max} - \frac{t_0''}{1-\nu}\left(\frac{\pi}{2} - T\right)^{1-\nu} \tag{7.13}$$
$$a \to t_0'' \left[(1-\nu)\left(\frac{t_{\max} - t}{t_0''}\right)\right]^{-1/\sqrt{3}}$$

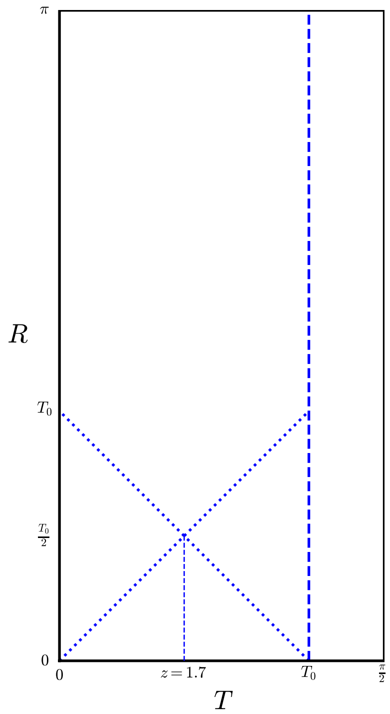
7.5 Table of parameters for selected z values

	z	H/H_0	q	w	Ω
$z \gg 1$	$\gg 1$	$0.75\ z^{1.7}$	0.73	0.16	1.0
	1000.	120000.	0.73	0.16	1.0
	100.	2200.	0.73	0.15	1.0
	10.	48.	0.74	0.15	1.0
$T = \frac{1}{2}T_0$	1.7	4.1	0.74	0.079	1.2
	1.0	2.4	0.69	0.021	1.3
$q = 0$	0.18	1.1	0.00	-0.33	1.5
$t = t_0$	0.00	1.0	-0.60	-0.61	1.5

7.6 Plot the cosmological parameters

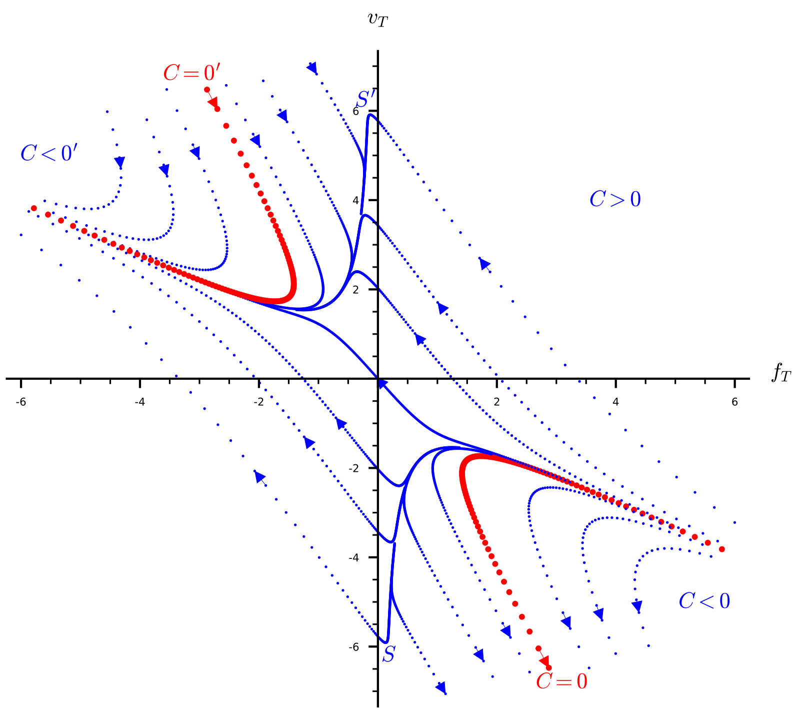


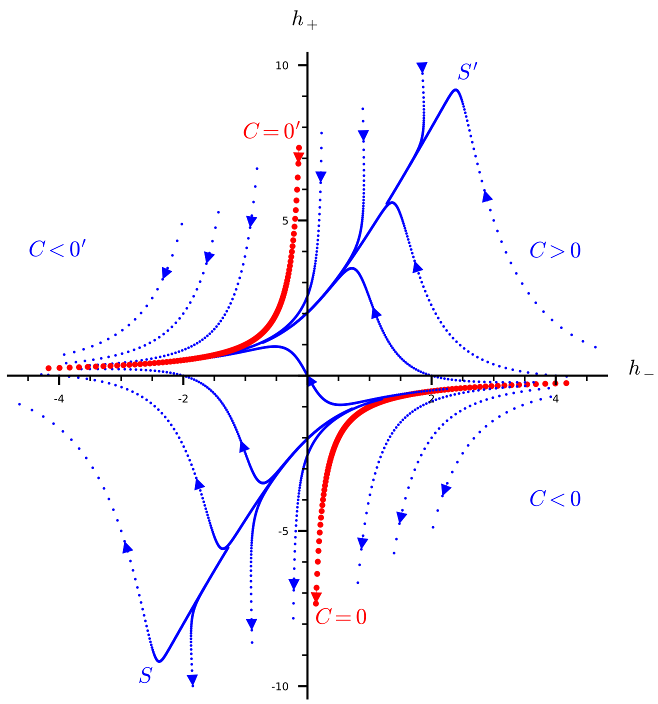
7.7 Plot the conformal diagram



8 Analysis of the real-time ode continued

8.1 Real-time phase portrait [SNB I-1]





8.2 Asymptotic behavior of the $C \neq 0$ solutions

Three asymptotic regions:

1. $h_+ \to \pm\infty, \quad |h_-| \leq |h_+|^{1-\epsilon}$
2. $h_- \to \pm\infty, \quad |h_+| \leq |h_-|^{1-\epsilon}$
3. $h_+, h_- \to \pm\infty, \quad |h_-| \sim |h_+|$

In region 1 we can approximate in the ode (6.3)

$$
\begin{gathered}
\partial_T h_+ \to -h_+^2 \qquad h_+ \to \frac{1}{T} \qquad T \to 0^\pm \\
f_T \to -\frac{\sqrt{3}}{2}\beta_- h_+ \to \frac{-\nu}{T} \qquad a^2 \to a_1^2 (T^2)^{-\nu} \qquad h_+ h_- = -1 + Ca^2 \to Ca_1^2 (T^2)^{-\nu} \\
h_- \to Ca_1^2 (T^2)^{-\nu} T \to 0
\end{gathered}
\tag{8.1}
$$

In region 2

$$
\begin{gathered}
\partial_T h_- \to -h_-^2 \qquad h_- \to \frac{1}{T} \qquad T \to 0^\pm \\
f_T \to \frac{\sqrt{3}}{2}\beta_+ h_- \to \frac{1+\nu}{T} \qquad a^2 \to a_2^2 (T^2)^{1+\nu} \qquad h_+ h_- = -1 + Ca^2 \to -1 + Ca_2^2 (T^2)^{1+\nu} \\
h_+ \to -T \to 0 \\
\partial_T h_+ \to -1 - T^2 + \lambda_+ C a_2^2 (T^2)^{1+\nu} + O(T^3) \\
h_+ \to -T - \frac{1}{3}T^3 + Ca_2^2 (T^2)^{1+\nu} T + O(T^4)
\end{gathered}
\tag{8.2}
$$

In region 3 the only asymptotic trajectories are $f_T \to 0$, $v_T \to \pm\infty$. The ode (3.4) is approximated

$$
\begin{gathered}
\partial_T f_T = -2(f_T v_T + 1) \qquad \partial_T v_T = 4 f_T v_T + 3 \\
f_T v_T + 1 \to 0 \qquad \partial_T v_T \to -1 \\
v_T \to -T \qquad f_T \to \frac{1}{T} \qquad T \to \pm\infty
\end{gathered}
\tag{8.3}
$$

In summary, there are three reflection pairs of asymptotic trajectories,

$$
\begin{array}{llll}
1. & T \to 0^\pm & h_+ \to \frac{1}{T} & h_- \to Ca_1^2 (T^2)^{-\nu} T \\
2. & T \to 0^\pm & h_- \to \frac{1}{T} & h_+ \to -T - \frac{1}{3}T^3 + Ca_2^2 (T^2)^{1+\nu} T \\
3. & T \to \pm\infty & f_T \to \frac{1}{T} & v_T \to -T
\end{array}
\tag{8.4}
$$

Every trajectory asymptotes as type 1 or type 2. Except for the separatrix S and its reflection S' every trajectory asymptotes at both ends as type 1 or type 2. S and S' each asymptote at one end as type 2.

8.3 Non-analyticity of $C \neq 0$ solutions

Try to analytically continue to imaginary time, replacing $T \to e^{-i\theta}\tau$ with θ ranging from 0 to $\pi/2$, $f_T \to e^{i\theta} f_\tau$, $v_T \to e^{i\theta} v_\tau$, $h_\pm \to e^{i\theta} h_{E\pm}$. The type 1 and 2 asymptotic behaviors become

$$\begin{array}{lllll} 1. & \tau \to 0^{\pm} & h_{E+} \to \dfrac{1}{\tau} & h_{E-} \to e^{(2\nu-2)i\theta} C a_1^2 (\tau^2)^{-\nu} \tau \\ 2. & \tau \to 0^{\pm} & h_{E-} \to \dfrac{1}{\tau} & h_{E+} \to \tau - \dfrac{1}{3}\tau^3 + e^{-(2\nu+4)i\theta} C a_2^2 (T^2)^{1+\eta} T \end{array} \tag{8.5}$$

At $\theta = \pi/2$ this is

$$\begin{array}{lllll} 1. & \tau \to 0^{\pm} & h_{E+} \to \dfrac{1}{\tau} & h_{E-} \to -e^{\nu i\pi} C a_1^2 (\tau^2)^{-\nu} \tau \\ 2. & \tau \to 0^{\pm} & h_{E-} \to \dfrac{1}{\tau} & h_{E+} \to \tau - \dfrac{1}{3}\tau^3 + e^{-\nu i\pi} C a_2^2 (T^2)^{1+\eta} T \end{array} \tag{8.6}$$

but $h_{E\pm} = f_\tau + \beta_\pm v_\tau$ should be real. So none of the $C \neq 0$ real-time solutions can be analytic continuations of euclidean signature solutions.

8.4 Asymptotic behavior of the separatrices S, S' [SNB I-1]

The other ends of the separatrices S, S' have type 3 asymptotic behavior. They have expansions at large T

$$T \to \pm\infty \qquad f_T = \sum_{m=0} f_m T^{-2m-1} \qquad v_T = -T + \sum_{m'=0} v_{m'} T^{-2m'-1} \qquad f_0 = 1 \tag{8.7}$$

The coefficients f_m, $v_{m'}$ are determined recursively by the ode (3.4). For $m \geq 0$

$$\begin{aligned} v_m &= -2f_m + \frac{1}{2m+1} \sum_{m'+m''=m} f_{m'} f_{m''} \\ f_{m+1} &= -\frac{1}{2}(2m+1) f_m + \sum_{m'+m''=m} f_{m'} \left(f_{m''} + v_{m''}\right) \end{aligned} \tag{8.8}$$

Numerical calculation for large m gives

$$\begin{array}{c} \dfrac{-f_m}{f_{m-1}}, \quad \dfrac{-v_m}{v_{m-1}} \quad \to \quad e^{0.5/m}\, m \\ (-1)^m f_m \to 0.22\, m^{1/2} m! \qquad (-1)^{m+1} v_m \to 0.44\, m^{1/2} m! \end{array} \tag{8.9}$$

so the expansions (8.7) are not convergent.

8.5 Numerical calculation of the invariant c_S [SNB I-2]

Calculate

$$\partial_T h_- = -1 - h_-^2 + \lambda_-(h_+ h_- + 1) = -2\nu(1 + h_- f_T) \tag{8.10}$$

Re-write using $f_T = \partial_T a / a$ and $2\nu(1+\nu) = 1$

$$(1+\nu)\partial_T h_- + h_- \frac{\partial_T a}{a} + +1 = 0 \tag{8.11}$$

Away from $h_- = 0$ this is

$$\partial_T \left[(h_-^2)^{1+\nu} a^2\right] + \frac{2}{h_-} \left[(h_-^2)^{1+\nu} a^2\right] = 0 \tag{8.12}$$

For the trajectories that behave as $h_- \sim T^{-1}$ for $T \to 0$ define

$$s = \int_0^T \frac{2\,dT'}{h_-(T')} \tag{8.13}$$

so

$$\partial_T \left[(h_-^2)^{1+\nu} a^2 e^s \right] = 0 \tag{8.14}$$

so we can define a numerical invariant of the trajectory

$$c = Ca^2 (h_-^2)^{1+\nu} e^s = (h_+ h_- + 1)(h_-^2)^{1+\nu} e^s \qquad \partial_T c = 0 \tag{8.15}$$

In the limit $T \to 0$

$$c = \lim_{T\to 0} (h_+ h_- + 1)(h_-^2)^{1+\nu} \tag{8.16}$$

The separatrix S is one such trajectory. Its invariant c_S is calculated numerically:

1. Use the large T expansions (8.7) to get a point on S at large T.
2. Numerically integrate the ode backwards to get a point on S with $h_- \gg 1$.
3. Calculate s as a double expansion $s(x,y)$

$$x = \frac{1}{h_-^2} \qquad\qquad y = \frac{h_+ h_- + 1}{h_-^2} = x + \frac{h_+}{h_-} \tag{8.17}$$

   Substitute s in (8.15) to get c_S.

Calculate

$$\begin{aligned} \partial_T h_- &= -1 - h_-^2 + \lambda_-(h_+ h_- + 1) \\ &= -1 - x^{-1} + x^{-1}\lambda_- y \\ \partial_T h_+ &= -1 - h_+^2 + \lambda_+(h_+ h_- + 1) = -1 - \left(h_- y - h_-^{-1}\right)^2 + \lambda_+ x^{-1} y \\ &= -1 - x^{-1}y^2 + 2y - x + \lambda_+ x^{-1} y \end{aligned} \tag{8.18}$$

$$\begin{aligned} h_- \partial_T x &= \frac{-2}{h_-^2} \partial_T h_- = -2x \left[-1 - x^{-1} + x^{-1}\lambda_- y \right] = 2\,(1 + x - \lambda_- y) \\ \partial_T (h_- y) &= \partial_T (h_+ + h_-^{-1}) \\ h_- \partial_T y &= \partial_T (h_- y) - y \partial_T h_- = \partial_T (h_+ + h_-^{-1}) - y \partial_T h_- = \partial_T h_+ - (x+y) \partial_T h_- \\ &= (-1 - x^{-1}y^2 + 2y - x + \lambda_+ x^{-1} y) - (x+y)(-1 - x^{-1} + x^{-1}\lambda_- y) \\ &= (1 + \lambda_+) x^{-1} y + (3 - \lambda_-) y - (1 + \lambda_-) x^{-1} y^2 \\ &= 2x^{-1} y \; [(2+\nu) + (1+\nu)x - (1-\nu)y] \end{aligned} \tag{8.19}$$

so

$$\begin{aligned} \frac{1}{2} h_- \partial_T x &= \partial_s x = 1 + x - \lambda_- y \\ \frac{1}{2} h_- \partial_T y &= \partial_s y = x^{-1} y \; [(2+\nu) + (1+\nu)x - (1-\nu)y] \end{aligned} \tag{8.20}$$

$(2+\nu)\lambda_- = (1-\nu)$ so

$$\begin{aligned} xy^{-1}\partial_s y - (2+\nu)\partial_s x &= -x \\ \partial_s \left[\ln y - (2+\nu) \ln x + s \right] &= 0 \\ y x^{-2-\nu} e^s = c \qquad \partial_s c &= 0 \end{aligned} \tag{8.21}$$

Consider e^s as a function $e^s(x,y)$ defined by

$$e^s(0,0) = 1 \qquad e^s = \partial_s e^s = \partial_s x \partial_x e^s + \partial_s y \partial_y e^s \tag{8.22}$$

which is

$$x\, e^s = (1 + x - \lambda_- y) x \partial_x e^s + [(2+\nu) + (1+\nu)x - (1-\nu)y]\, y \partial_y e^s \tag{8.23}$$

At $y = 0$ the solution is

$$e^s(x,0) = 1 + x \tag{8.24}$$

Expand around $x = y = 0$

$$e^s = \sum_{j,k\geq 0} \mathcal{S}_{j,k} x^j y^k \tag{8.25}$$

$$\begin{aligned} x \sum_{j,k\geq 0} \mathcal{S}_{j,k} x^j y^k &= (1 - \lambda_- y + x) \sum_{j,k\geq 0} \mathcal{S}_{j,k} j x^j y^k \\ &\quad + [(2+\nu) - (1-\nu)y + (1+\nu)x] \sum_{j,k\geq 0} \mathcal{S}_{j,k} k x^j y^k \end{aligned} \tag{8.26}$$

$$\begin{aligned} \mathcal{S}_{j-1,k} &= j\mathcal{S}_{j,k} - \lambda_- j \mathcal{S}_{j,k-1} + (j-1)\mathcal{S}_{j-1,k} \\ &\quad + (2+\nu)k\mathcal{S}_{j,k} - (1-\nu)(k-1)\mathcal{S}_{j,k-1} + (1+\nu)k\mathcal{S}_{j-1,k} \end{aligned}$$

So we get a recursion formula

$$[j + (2+\nu)k]\mathcal{S}_{j,k} = [2 - j - (1+\nu)k]\mathcal{S}_{j-1,k} + [\lambda_- j + (1-\nu)(k-1)]\mathcal{S}_{j,k-1} \tag{8.27}$$

For $j = 0$

$$\begin{aligned} (2+\nu)k\mathcal{S}_{0,k} &= (1-\nu)(k-1)\mathcal{S}_{0,k-1} \\ \mathcal{S}_{0,k} &= 0 \qquad k \geq 1 \end{aligned} \tag{8.28}$$

so the recursion can start from

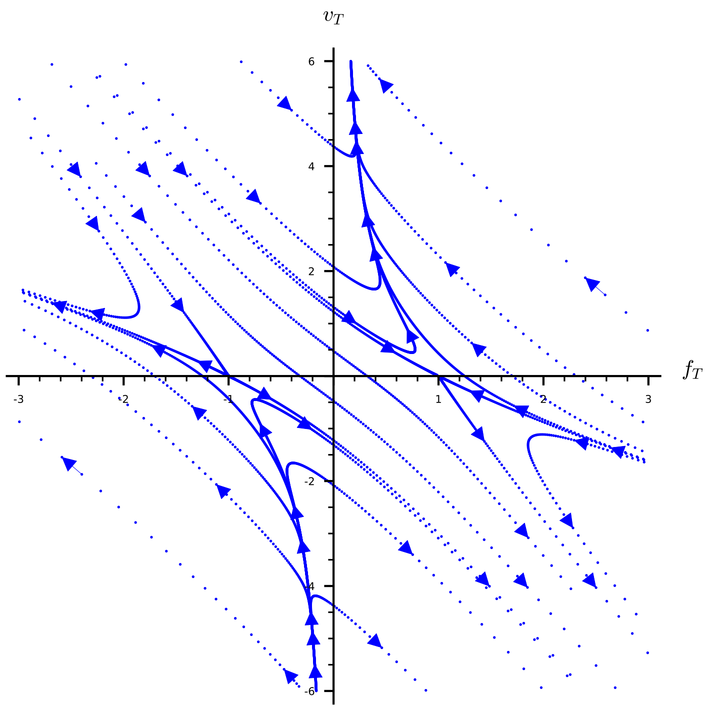
$$\mathcal{S}_{0,0} = 1 \qquad \mathcal{S}_{1,0} = 1 \qquad \mathcal{S}_{j,0} = 0 \quad j \geq 2 \qquad \mathcal{S}_{0,k} = 0 \quad k \geq 1 \tag{8.29}$$

The expansion might be cut off at $O(x^N)$ according to

$$\begin{gathered} y = O(x^{2+\nu}) \qquad x^j y^k = O(x^{j+k(2+\nu)}) = O(x^N) \qquad j + k(2+\nu) \leq N \\ j \leq N \quad k \leq N/(2+\nu) \end{gathered} \tag{8.30}$$

9 Euclidean signature phase portrait [SNB I-3]

The phase portrait of the euclidean signature ode (2.4) is shown for completeness. It is not used in the paper.



The ode

$$\partial_\tau f_\tau = -2 f_\tau v_\tau - 2 f_\tau^2 + 2 \qquad \partial_\tau v_\tau = 4 f_\tau v_\tau + 3 f_\tau^2 - 3 \tag{9.1}$$

is regarded as a dynamical system. A solution is a trajectory in the f_τ, v_τ plane parametrized by τ. The ode is invariant under the time-reflection symmetry

$$\tau \to -\tau \qquad (f_\tau, v_\tau) \to (-f_\tau, -v_\tau) \tag{9.2}$$

There are two fixed points, at $(1,0)$ and at its reflection $(-1,0)$. The linearized ode at $(1,0)$ is

$$\partial_\tau \begin{pmatrix} \Delta f_\tau + \Delta v_\tau \\ 3\Delta f_\tau + \Delta v_\tau \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \Delta f_\tau + \Delta v_\tau \\ 3\Delta f_\tau + \Delta v_\tau \end{pmatrix} \tag{9.3}$$

so the fixed point has an unstable manifold tangent to $\Delta v_\tau = -3\Delta f_\tau$ and a stable manifold tangent to $\Delta v_\tau = -\Delta f_\tau$.

The analytic continuation of the $C = 0$ solution (6.7) from T to $\tau = iT$ is

$$f_\tau = -\frac{\cosh 2\tau + \sqrt{3}}{\sinh 2\tau} \qquad v_\tau = \frac{\sqrt{3}}{\sinh 2\tau} \tag{9.4}$$

In the phase portrait this is the union of two trajectories. The portion $-\infty \le \tau < 0$ is the unstable trajectory leaving the fixed point $(1,0)$ going into the lower-right quadrant. The portion $0 < \tau \le \infty$ is the time-reflection, the stable trajectory of the fixed point $(-1,0)$ entering from the upper-left quadrant. The two trajectories are joined at the point at ∞ in the (f_τ, v_τ) plane at $\tau = 0$

$$(f_\tau, v_\tau) \to \left(-\frac{1+\sqrt{3}}{2\tau}, \frac{\sqrt{3}}{2\tau} \right) \qquad \tau \to 0^\pm \tag{9.5}$$

The phase portrait is calculated using numerical integration from initial conditions at large f_τ. The trajectories which reach large f_τ are given asymptotically by expansions around $\tau = 0$

$$f_\tau = \tau^{-1} \sum_{m,n=0} \beta^m (\tau^2)^{m\alpha/2} \tau^{2n} \mathbf{f}_{m,n} \qquad v_\tau = \tau^{-1} \sum_{m,n=0} \beta^m (\tau^2)^{m\alpha/2} \tau^{2n} \mathbf{v}_{m,n} \tag{9.6}$$

where α is a positive real number to be determined and β is a real number which parametrizes the trajectory and ranges over some interval around 0. The coefficients $\mathbf{f}_{m,n}$ and $\mathbf{v}_{m,n}$ are given by recursion relations obtained by substituting in the ode. For $(m,n) = (0,0)$

$$\begin{aligned} -\mathbf{f}_{0,0} &= \mathbf{f}_{0,0} \left(-2\mathbf{v}_{0,0} - 2\mathbf{f}_{0,0}\right) \\ -\mathbf{v}_{0,0} &= \mathbf{f}_{0,0} \left(4\mathbf{v}_{0,0} + 3\mathbf{f}_{0,0}\right) \end{aligned} \qquad \begin{pmatrix} \mathbf{f}_{0,0} \\ \\ \mathbf{v}_{0,0} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \epsilon\frac{\sqrt{3}}{2} \\ \\ -\epsilon\frac{\sqrt{3}}{2} \end{pmatrix} \qquad \epsilon = \pm 1 \tag{9.7}$$

Then the recursion relation for $(m,n) \neq (0,0)$ is

$$\begin{gathered} A_{m,n} \begin{pmatrix} \mathbf{f}_{m,n} \\ \mathbf{v}_{m,n} \end{pmatrix} = \delta_{m,0}\delta_{n,1} \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 3 & 4 \end{pmatrix} \sum_{\substack{m'+m''=m \\ n'+n''=n \\ \neq(0,0),(m,n)}} \mathbf{f}_{m',n'} \begin{pmatrix} \mathbf{f}_{m'',n''} \\ \mathbf{v}_{m'',n''} \end{pmatrix} \\ A_{m,n} = \begin{pmatrix} \alpha m + 2n - 1 + 2\mathbf{v}_{0,0} + 4\mathbf{f}_{0,0} & 2\mathbf{f}_{0,0} \\ -4\mathbf{v}_{0,0} - 6\mathbf{f}_{0,0} & \alpha m + 2n - 1 - 4\mathbf{f}_{0,0} \end{pmatrix} \\ \det A_{m,n} = (\alpha m + 2n + 1)(\alpha m + 2n - 3 - \epsilon\sqrt{3}) \end{gathered} \tag{9.8}$$

The case $(m,n) = (1,0)$ requires

$$\det A_{1,0} = 0 \qquad \alpha = 3 + \epsilon\sqrt{3} \qquad \begin{pmatrix} \mathbf{f}_{1,0} \\ \mathbf{v}_{1,0} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 - \epsilon\sqrt{3} \end{pmatrix} \tag{9.9}$$

Then $\det A_{m.n} \neq 0$ for $(m,n) \neq (1,0)$ so the recursion relation determines all the remaining coefficients.

10 $v_\mu = \partial_\mu \chi \iff \chi$ is the dilaton field

If $v_\mu = \partial_\mu \chi$, the fixed point equation is

$$R_{\mu\nu} = 2\nabla_\mu \nabla_\nu \chi \tag{10.1}$$

and χ satisfies

$$\nabla_\mu \nabla^\mu \chi + 2\nabla_\mu \chi \nabla^\mu \chi - C' = 0 \tag{10.2}$$

These are the equations of motion of the dilaton action

$$S = \int d^d x \sqrt{g}\, e^{2\chi} (-R - 4\nabla_\mu \chi \nabla^\mu \chi - 2C') \tag{10.3}$$

So if $v_\mu = \partial_\mu \chi$ then χ is the dilaton field. But v^μ is not necessarily a gradient in general.

To derive the dilaton equations of motion, first vary $\chi \to \chi + \delta\chi$

$$\begin{aligned} \delta S &= \int d^d x \sqrt{g}\, e^{2\chi} \left[2\delta\chi(-R - 4\nabla_\mu \chi \nabla^\mu \chi - 2C') - 8\nabla_\mu \chi \nabla^\mu \delta\chi\right] \\ &= \int d^d x \sqrt{g}\, e^{2\chi} \delta\chi \left[-2R - 8\nabla_\mu \chi \nabla^\mu \chi - 4C' + 8e^{-2\chi} \nabla^\mu (e^{2\chi} \nabla_\mu \chi)\right] \\ &= \int d^d x \sqrt{g}\, e^{2\chi} \delta\chi \left[-2R - 8\nabla_\mu \chi \nabla^\mu \chi - 4C' + 4e^{-2\chi} (\nabla^\mu \nabla_\mu) e^{2\chi}\right] \end{aligned} \tag{10.4}$$

Vary $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ using

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}\left[\nabla_\sigma \nabla_\mu \delta g^\sigma_\nu + \nabla_\sigma \nabla_\nu \delta g^\sigma_\mu - \nabla^\sigma \nabla_\sigma \delta g_{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^\sigma_\sigma\right] \\ g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\mu \nabla_\nu \delta g^{\mu\nu} - g_{\mu\nu}\nabla^\sigma \nabla_\sigma \delta g^{\mu\nu}\end{aligned} \tag{10.5}$$

$$\begin{aligned}\delta S &= \int d^dx\,\sqrt{g}\,e^{2\chi}\left[\frac{1}{2}\delta g^{\mu\nu} g_{\mu\nu}(-R - 4\nabla_\sigma\chi\nabla^\sigma\chi - 2C') - \delta g^{\mu\nu}(-R_{\mu\nu} - 4\nabla_\mu\chi\nabla_\nu\chi) - g^{\mu\nu}\delta R_{\mu\nu}\right] \\ &= \int d^dx\,\sqrt{g}\,e^{2\chi}\left[\frac{1}{2}\delta g^{\mu\nu} g_{\mu\nu}(-R - 4\nabla_\sigma\chi\nabla^\sigma\chi - 2C') - \delta g^{\mu\nu}(-R_{\mu\nu} - 4\nabla_\mu\chi\nabla_\nu\chi)\right. \\ &\qquad \left. + (-\nabla_\mu\nabla_\nu\delta g^{\mu\nu} + g_{\mu\nu}\nabla^\sigma\nabla_\sigma\delta g^{\mu\nu})\right] \\ &= \int d^dx\,\sqrt{g}\,e^{2\chi}\delta g^{\mu\nu}\left[R_{\mu\nu} + 4\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}(R + 4\nabla_\sigma\chi\nabla^\sigma\chi + 2C')\right. \\ &\qquad \left. + e^{-2\chi}\left(-\nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^\sigma\nabla_\sigma\right)e^{2\chi}\right]\end{aligned} \tag{10.6}$$

So the equations of motion are

$$\begin{aligned}0 &= R + 4\nabla_\mu\chi\nabla^\mu\chi + 2C' - 2e^{-2\chi}(\nabla^\mu\nabla_\mu)e^{2\chi} \\ 0 &= R_{\mu\nu} + 4\nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}(R + 4\nabla_\sigma\chi\nabla^\sigma\chi + 2C') \\ &\quad + e^{-2\chi}\left(-\nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^\sigma\nabla_\sigma\right)e^{2\chi}\end{aligned} \tag{10.7}$$

Substituting the first equation of motion in the second, the second becomes

$$0 = R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\chi \tag{10.8}$$

so

$$R = 2\nabla^\mu\nabla_\mu\chi \tag{10.9}$$

Substituting for R, the first equation of motion becomes

$$0 = \nabla_\mu\nabla^\mu\chi + 2\nabla_\mu\chi\nabla^\mu\chi - C' \tag{10.10}$$

So the dilaton equations of motion are

$$R_{\mu\nu} = 2\nabla_\mu\nabla_\nu\chi \qquad \nabla^\mu\nabla_\mu\chi + 2\nabla^\mu\chi\nabla_\mu\chi - C' = 0 \tag{10.11}$$