On the Procedure of Mass Voting on a Slate of Positions and the Significant Number

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1 Preliminaries

To begin, we need to establish notation for the majority of a number:

Definition 1. Given a positive integer n, the *majority of* n is the smallest positive integer $M[n] < n$ such that $\frac{n}{2} < M[n]$. Explicitly,

$$M[n] = \begin{cases} \frac{n}{2} + 1 & n \text{ is even} \\ \frac{n-1}{2} + 1 & n \text{ is odd.} \end{cases}$$

> **Example.** The majority of $n = 7$ is $M[7] = \frac{7-1}{2} + 1 = \frac{6}{2} + 1 = 3 + 1 = 4$.

The first few majorities are summarized in the table below:

n	1	2	3	4	5	6	7	8	9	10
$M[n]$	1	2	2	3	3	4	4	5	5	6

Definition 2. A *slate* is a list of positions and the the candidate chosen for each position. Generally, we will fill in the other candidates for each position underneath the candidate chosen.

2 Optimization

Now we can discuss abstractly the idea of *optimizing* a slate of candidates. Consider a slate with 3 positions and 7 candidates, names notated with capital letters, A, B, C, D, E, F, and G. Imagine that for each position, each candidate, X, was given a number $\#_p$, on a scale from 0 to 10, for desirability for position number p. In this example, we will notate this,

$$A(\#_p)$$

for person A receiving a ranking of $\#_p$ for position number p. Here is what the blank slate will look like:

position 1	position 2	position 3
$A(\#_1)$	$A(\#_2)$	$A(\#_3)$
$B(\#_1)$	$B(\#_2)$	$B(\#_3)$
$C(\#_1)$	$C(\#_2)$	$C(\#_3)$
$D(\#_1)$	$D(\#_2)$	$D(\#_3)$
$E(\#_1)$	$E(\#_2)$	$E(\#_3)$
$F(\#_1)$	$F(\#_2)$	$F(\#_3)$
$G(\#_1)$	$G(\#_2)$	$G(\#_3)$

To optimize the desirability of the three candidates as a whole, you must maximize the sum of the rankings for each position. That is, you must choose the candidates in such a way, so that the sum of the rankings that each winning candidate received for the position they were awarded, is maximized. In this example, you would choose $X(\#_1)$, $Y(\#_2)$, and $Z(\#_3)$ for positions 1, 2, and 3, respectively so the sum

$$\#_1 + \#_2 + \#_3$$

is maximum. We will call this sum the *score* of the slate.

It is tempting to simply survey each column and choose the person with the highest number in that column for that position. This however, is overlooking quite a big issue. Consider the reduced example slate:

position 1	position 2	position 3
$A(9)$	$A(2)$	$A(8)$
$B(8)$	$B(4)$	$B(5)$
$C(4)$	$C(9)$	$C(3)$

If you were to choose,

position 1	position 2	position 3
$A(9)$	$C(9)$	$A(8)$

obviously you cannot choose A for two positions so that choice produces an immediate problem. So, supposed then you choose to give position 3 to B because it was the runner up in that column and $A(9)$ is higher than $A(8)$. This would leave you with the slate:

position 1	position 2	position 3
$A(9)$	$C(9)$	$B(5)$

which has a score of $9 + 9 + 5 = 23$.

But consider the slate:

position 1	position 2	position 3
$B(8)$	$C(9)$	$A(8)$

Which has a score of $8 + 9 + 8 = 25$.

This simple example shows that clearly, the idea of optimized slating is not an simple one. With more complicated and larger slates, the solution is not so obvious. I will not discuss further methods for such optimization, because that is not the focus of this paper. However, I think it necessary to discuss in order to impress upon someone who is attempting to construct a slate, the delicacy required.

3 The Significant Number

The idealized notion of cleanly ranking candidates suggested in the last section might not be possible or even feasible in a real election, so here I will discuss how the voting procedure can be optimized in order to reduce conflict and time so that the slate can be found subjectively.

We will consider an election for board members, where members of the current board are given the opportunity to vote for the positions that they themselves are not running for. I will notate board member candidates with capital letters and non-board candidates with lowercase letters.

In an effort to save time and minimize the likelihood of offending anyone, the slate, in completion, will be voted on by the board. If the slate gets rejected, the positions will be scrutinized one by one, where only the voting member of the board for that position are present. After a discussion is had for each position, the next optimized board is proposed and the process restarts.

Now we are faced with a problem: since the number of people who have a vote for each position is different, how can we interpret a majority? As a reminder only board members who are not a candidate for a position are allowed to vote for that position.

In this section we will discuss the *significant number*, the minimum number of people that must object to the slate in order for it to be rejected. One might mistakenly say that this number is the majority, but consider the following example slate and a board of 7 members:

position 1	position 2	position 3
A	B	C
D	E	F
G	h	B
F	j	h
j	k	i
		k

There are 7 board members, but for position 1, the number of eligible voting members is only $7 - 4 = 3$ since 4 of the 7 board members are themselves running for that position. This means that for position 1, the majority is $M[3] = 2$.

So, if 2 people object to the slate, even though 2 is not the majority of 7, there is a position for which a vote of 2 has the majority, so the slate as a whole must be rejected. If it is not, then the 4 members of the board who do not get a vote for position 1, are in affect, getting a vote, because they are able to suppress the only people who can overrule them.

Definition 3. In general, the *significant number*, S, is the minimum of the position-wise voting pool majorities and is the minimum number of votes needed to reject the slate.

The calculation of S is not difficult.

Let v_p be the number of eligible voters for position p. v_p is always the total number of board members minus the number of board members who are running for position p. From our example,

$v_1 = 7 - 4 = 3$, because A, D, G, and F are on the board,

$v_2 = 7 - 2 = 5$, because B and E are on the board, and

$v_3 = 7 - 3 = 4$ since C, F, and B are on the board.

This means we have voting pools of size $v_1 = 3$, $v_2 = 5$, and $v_3 = 4$, each with majorities

$M[v_1] = M[3] = 2$,

$M[v_2] = M[5] = 3$, and

$M[v_3] = [4] = 2$.

The minimum of the majorities is 2, so the significant number is $S = 2$.

In general,

$$S = \min\{M[v_p] \ : \text{for all positions, } p \ \}.$$

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