All work on this lab should be the collective effort of all group members. Technology allowed on this lab includes: Desmos (`https://www.desmos.com/calculator`) and an approved TI calculator. This lab has 7 questions for a total of 8 points.

1. (**Lorentz contraction**) In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v, the length will appear to be

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$

This equation is the **Lorentz contraction** formula. Here, c is the speed of light in a vacuum, which is about 3×10^8m/sec.

(a) (4 points) What happens to L as v increases?

Solution: As v increase, the value of L will approach zero. This means that the observable length of the rocket will seem to go to zero.

(b) (4 points) Find $\lim_{v \to c^-} L$.

(b) **0**

(c) Why was the left-hand limit needed in the limit?

Solution: The significance of the left-hand limit is that there is no known particle – let alone rocket – that can travel faster than the speed of light. Therefore, you can only approach c from numbers below it, i.e. from the left.

2. Let $f(x) = \sqrt{x+1}$. Find the largest $\delta > 0$ such that $f(x) \in (2 - \varepsilon, 2 + \varepsilon)$, when $\varepsilon = 0.1$.

Solution: Since we are given that $\varepsilon = 0.1$, we know that $\sqrt{x+1} \in (2 - 0.1, 2 + 0.1)$ or in the interval $(1.9, 2.1)$. Using this information, which is an epsilon-band around 2, we can find a punctured delta interval, that should be centered at 3, to correspond to this epsilon-band. Begin by solving for the lower epsilon value and the lower epsilon value.

Lower x-value

$$\sqrt{x+1} = 1.9$$
$$x + 1 = 3.61$$
$$x = 2.61$$

Upper x-value

$$\sqrt{x+1} = 2.1$$
$$x + 1 = 4.41$$
$$x = 3.41$$

This means we have a the punctured delta interval of $(2.61, 3) \cup (3, 3.41)$. The distance between the two intervals gives two delta values $\delta_1 = 0.39, \delta_2 = 0.41$. To ensure the values stay within the given epsilon-band, we need to choose the minimum of the two. So, we say choose $\delta = 0.39$.

3. Use epsilon-delta proofs of a limit to prove that $\lim_{x \to -11} \frac{x^2+6x+5}{x+5} = -10$.

Solution: *Proof.* Given $\varepsilon > 0$, choose $\delta = \boxed{\varepsilon}$, such that if $0 < |x+11| < \delta$, then

$$\left| \frac{x^2+6x+5}{x+5} + 10 \right| = \left| \frac{(x+5)(x+1)}{(x+5)} + 10 \right| = |x+1+10| = |x+11| < \varepsilon.$$

Therefore, when $\delta = \varepsilon$, if $0 < |x+11| < \delta$, then $\left| \frac{x^2+6x+5}{x+5} + 10 \right| < \varepsilon$.

4. Consider the function $g(x) = \begin{cases} 0, & x \le 0 \\ \sin\left(\frac{1}{x}\right), & x > 0 \end{cases}$.

(a) Does $\lim_{x \to 0^+} g(x)$ exist? If so, what is it? If not, why not?

Solution: No, the limit as x approaches zero from the right does not exist because the function $\sin(1/x)$ oscillates between -1 and 1.

(b) Does $\lim_{x \to 0^-} g(x)$ exist? If so, what is it? If not, why not?

Solution: Yes, the limit as x approaches zero from the left does exist because the function is just $y = 0$, for all values to the left of zero, but not at $x = 0$.

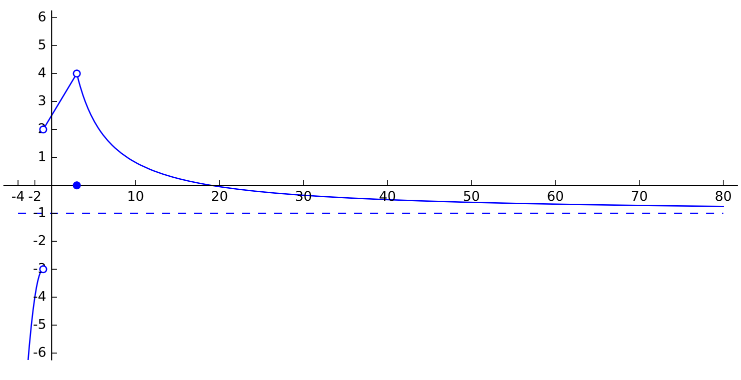
(c) Does $\lim_{x \to 0} g(x)$ exist? If so, what is it? If not, why not?

Solution: No, since the left and right limits are not equal, this limit does not exist.

5. Graph a function that meets the following criteria:

- $\lim_{x \to -\infty} h(x) = -\infty$,
- $\lim_{x \to -1^-} h(x) = -3$,
- $\lim_{x \to -1^+} h(x) = 2$,
- $\lim_{x \to 3} h(x) = 4$,
- $h(3) = 0$,
- $\lim_{x \to \infty} h(x) = -1$.

Solution: This is one particular example. There are many that would work.



6. Each of the following statements are false. Find a counterexample to each of the statements. Your counterexample may take many different forms, i.e. you may come up with an equation, provide a graph, or give a detailed explanation.

(a) For any function $f(x)$, $\lim_{x \to c} f(x) = f(c)$.

Solution: This is only true for **continuous** functions. An example could be $\lim_{x\to 0} \frac{\sin(x)}{x}$. We know, from class, that this limit is 1, but $\frac{\sin(0)}{0}$ is indeterminate.

(b) $\lim_{x\to\infty} \sin(x) = 0$ because $\frac{-1+1}{2} = 0$.

Solution: As $x \to \infty$ the value of $\sin(x)$ never approaches a single value. It alternates between -1 and 1. This means that the limit does not exist.

(c) All limits of the form $\infty - \infty$ must go to zero.

Solution: Counterexample: $\lim_{x\to\infty} (\sqrt{x} - x)$ has the indeterminate form $\infty - \infty$, but the actual limit value is $-\infty$.

(d) If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to L_1} g(x) = L_2$, then $\lim_{x \to c} (g \circ f)(x) = L_2$.

Solution: I'm still thinking about this one...

7. Calculate the following limits using algebra.

(a) $\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x}$

Solution:

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} &= \frac{(x-2)(x-2)}{x(x+7)(x-2)}\\ &= \lim_{x\to 2}\frac{x-2}{x(x+7)}\\ &= \frac{2-2}{2(2+7)}\\ &= 0\end{aligned}$$

(b) $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

Solution:

$$\begin{aligned}\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}} &= \lim_{x\to 1}\frac{1-x}{(1-x)(1+\sqrt{x})}\\ &= \lim_{x\to 1}\frac{1}{1+\sqrt{x}}\\ &= \frac{1}{1+\sqrt{1}}\\ &= \frac{1}{2}\end{aligned}$$

(c) $\lim_{x\to\infty} \frac{x^7-11x^3-5x-2}{3x-2x^2-17x^{11}+12}$

Solution: Since this is a rational functions where the power of the denominator is larger than the numerator, this limit goes to zero.

(d) $\lim_{x\to 0} \frac{\sin(7x)}{2x}$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(7x)}{2x} &= \lim_{x\to 0}\frac{7\sin(7x)}{7\cdot(2x)}\\ &= \lim_{x\to 0}\frac{7\sin(7x)}{2\cdot(7x)}\\ &= \frac{7}{2}\lim_{x\to 0}\frac{\sin(7x)}{(7x)}\\ &= \frac{7}{2}\cdot 1\\ &= \frac{7}{2}\end{aligned}$$

(e) $\lim_{x \to 0} \frac{3e^x - 3}{3e^{2x} + 9e^x - 12}$

Solution:

$$\begin{aligned}
\lim_{x \to 0} \frac{3e^x - 3}{3e^{2x} + 9e^x - 12} &= \lim_{x \to 0} \frac{3(e^x - 1)}{3(e^x + 4)(e^x - 1)} \\
&= \lim_{x \to 0} \frac{1}{(e^x + 4)} \\
&= \frac{1}{e^0 + 4} \\
&= \frac{1}{5}
\end{aligned}$$