MAT 271 Labs

This is a collection of labs to be used during the 2018 Summer semester.

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1 Section 0.5 – 1.2 Lab

1. Determine if the following statements are true or false. You must provide a justification for your answer.

   (a) TRUE / FALSE
   If a number is divisible by 6, then it is divisible by 3.

   (b) TRUE / FALSE
   For all real numbers x and y, $\frac{x}{y} = 0$ if and only if $x = 0$.

   (c) TRUE / FALSE
   For all real numbers y, there is a real number x such that $y = 2x + 4$.

   (d) TRUE / FALSE
   For all real numbers $x > 0$ and $y > 0$, if $x > y$, then $\dfrac{1}{x} < \dfrac{1}{y}$.

2. Consider the following implication:

   If x is divisible by 12, then x is divisible by 3.

   (a) Write the converse of the statement.

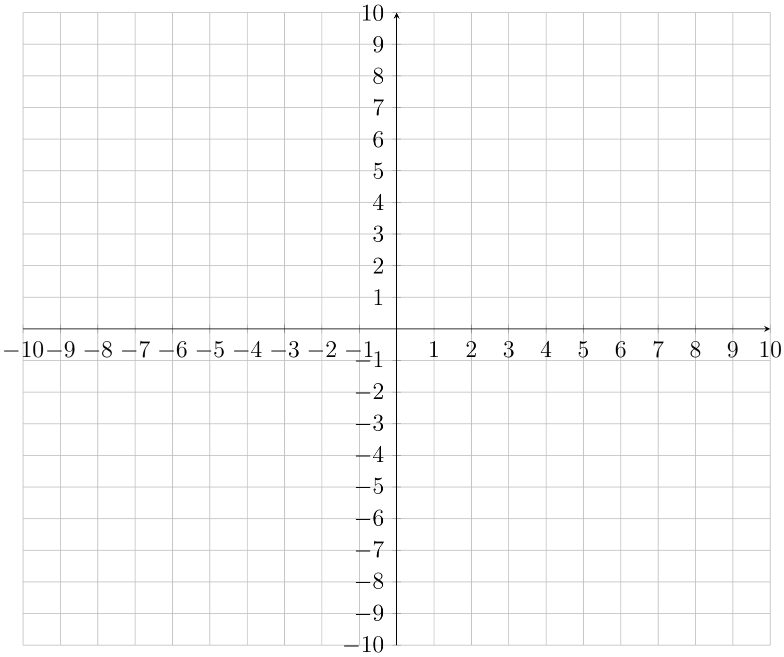
   The converse of the implication is TRUE / FALSE.

   (b) Write the contrapositive of the statement.

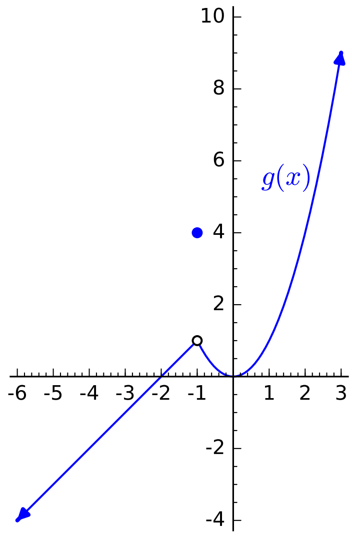
   The contrapositive of the implication is TRUE / FALSE.

3. Sketch the graph of a function that has the given limits and values. There is more than one correct answer.

- $\lim_{x \to -1} f(x) = 4$
- $\lim_{x \to \infty} f(x) = 4$
- $f(0) = 5$



4. Let $g(x)$ be the function graphed below. Determine the requested values.



(a) $\lim_{x \to -1} g(x) =$ ______

(b) $\lim_{x \to 1} g(x) =$ ______

(c) $g(-1) =$ ______

(d) $\lim_{x \to 2} g(x) =$ ______

(e) $\lim_{x \to -\infty} g(x) =$ ______

(f) $\lim_{x \to \infty} g(x) =$ ______

5. Give an argument – be it a table of values or a graph – that justifies the following limit:

$$\lim_{x\to 4} \frac{x-4}{(x+1)(x-4)} = \frac{1}{5}.$$

6. Let $h(x)$ be the following piecewise-defined function:

$$h(x) = \begin{cases} 3x-2 & \text{if } x > 2 \\ 0 & \text{if } x = 2 \\ x^2 & \text{if } x < 2 \end{cases}.$$

Using the graph of the function, determine the value of $\lim_{x\to 2} h(x)$.

6. ____________

7. Write the following limit using the $\varepsilon - \delta$ definition of a limit:

$$\lim_{x\to 3} \left(x^2 - 4\right) = 5.$$

8. Given that $\lim_{x\to 2} (2x+1) = 5$, use algebra to approximate the largest value of δ such that

if $x \in (2-\delta, 2) \cup (2, 2+\delta)$, then $f(x) \in (5-\varepsilon, 5+\varepsilon)$, where $\varepsilon = 0.01$.

2 Section 1.3 – 1.4 Lab

1. Write an epsilon-delta proof for each limit.

   (a) $\lim_{x \to 3}(2x - 3) = 3$

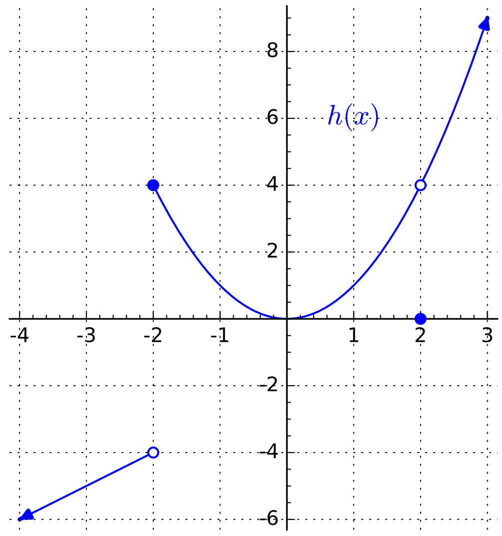
   (b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$

2. Consider the following piecewise-defined function:

$$f(x) = \begin{cases} 2x^2 + b, & \text{if } x \geq 1 \\ -x^3, & \text{if } x < 1 \end{cases}.$$

   Find the value of b such that $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x)$.

3. Let $h(x)$ be the function graphed below. Use the graph of h to answer questions about its continuity.



(a) What type of continuity does the graph of h have at $x = -2$? Explain your answer.

RIGHT / LEFT / BOTH / NEITHER

(b) What type of continuity does the graph of h have at $x = 2$? Explain your answer.

RIGHT / LEFT / BOTH / NEITHER

(c) What type of discontinuity does the graph of h have at $x = -2$?

JUMP / INFINITE / REMOVABLE / NONE OF THESE

(d) What type of discontinuity does the graph of h have at $x = 2$?

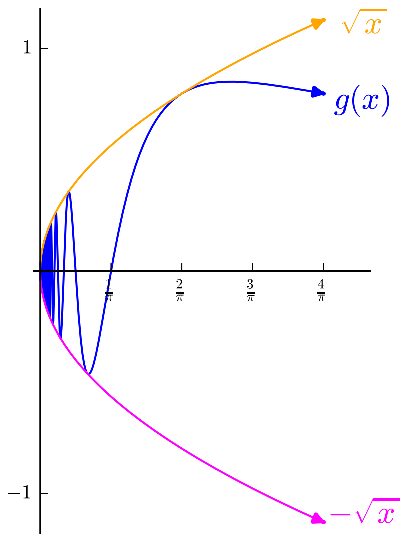
JUMP / INFINITE / REMOVABLE / NONE OF THESE

4. The **extreme value theorem** states the following:

If a real-valued function f is continuous in the closed and bounded interval $[a, b]$, then f must attain a maximum and a minimum, each at least once.

Does the function $f(x) = x^2 - 2x + 1$ satisfy the conditions of the extreme value theorem? Explain your answer. [Hint: For this problem, you do **not** have to worry about left-hand and right-hand continuity.]

5. (Looking Ahead) Consider the graph of the function $g(x) = \sqrt{x}\sin\left(\frac{1}{x}\right)$, graphed below.



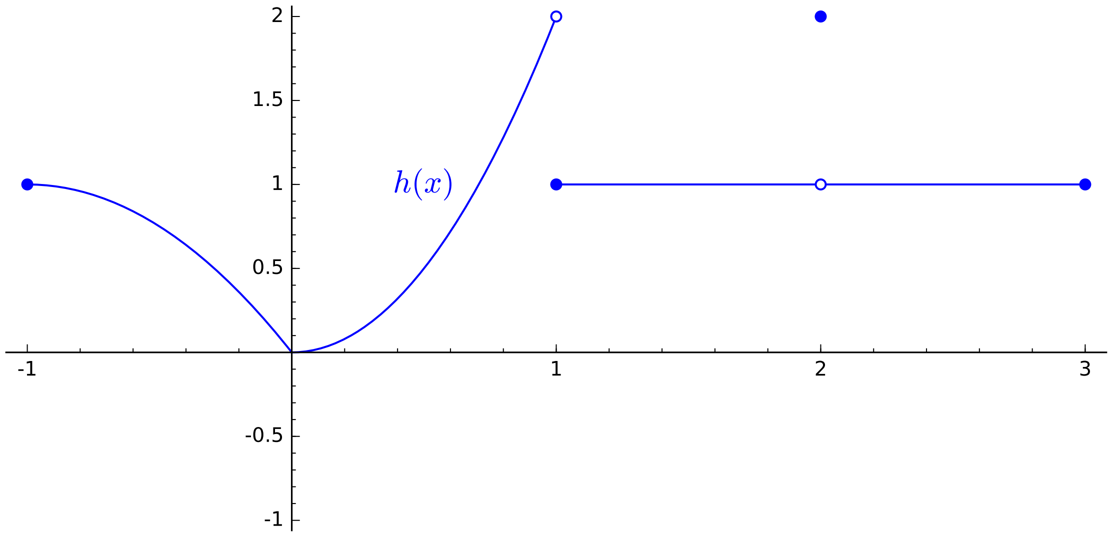
(a) Estimate the value of $\lim_{x \to 0^+} g(x)$.

(a) ______________

(b) Explain why $\lim_{x \to 0^-} g(x)$ does not exist.

(c) Use the domain of the function $g(x)$ to explain why this function is continuous on $(0, \infty)$, but not on $[0, \infty)$.

6. The complete graph of $h(x)$ is shown in the figure below. Answer TRUE / FALSE for each of the following questions regarding the graph of h.



(a) $\lim\limits_{x\to -1^-} h(x) = 1$ TRUE / FALSE

(b) $\lim\limits_{x\to 2} h(x) = 2$ TRUE / FALSE

(c) $\lim\limits_{x\to 1^+} h(x) = 1$ TRUE / FALSE

(d) $\lim\limits_{x\to 1^-} h(x) = 2$ TRUE / FALSE

(e) $\lim\limits_{x\to -1^-} h(x) = 0$ TRUE / FALSE

(f) $\lim\limits_{x\to c} h(x)$ exists at every c in the open interval $(-1, 1)$. TRUE / FALSE

(g) $\lim\limits_{x\to c} h(x)$ exists at every c in the open interval $(1, 3)$. TRUE / FALSE

(h) $\lim\limits_{x\to 3^+} h(x)$ does not exist. TRUE / FALSE

(i) $\lim\limits_{x\to 1} h(x)$ does not exist. TRUE / FALSE

(j) The domain of $h(x)$ is $[-1, 3]$. TRUE / FALSE

3 Section 1.5 – 1.6 Lab

1. (**Lorentz contraction**) In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v, the length will appear to be
$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$
This equation is the **Lorentz contraction** formula. Here, c is the speed of light in a vacuum, which is about 3×10^8m/sec.
   (a) What happens to the value of L as v increases?

   (b) Find $\lim\limits_{v\to c^-} L$.

   (b) ____________

   (c) Why was the left-hand limit needed in the limit from part (b)?

2. Let $f(x) = \sqrt{x+1}$. Find the largest $\delta > 0$ such that $f(x) \in (2-\varepsilon, 2+\varepsilon)$, when $\varepsilon = 0.1$.

3. Use an epsilon-delta proof to show that $\lim\limits_{x\to -11} \dfrac{x^2+6x+5}{x+5} = -10$.

4. Consider the function $g(x) = \begin{cases} 0, & x \leq 0 \\ \sin\left(\frac{1}{x}\right), & x > 0 \end{cases}$.

(a) Does $\lim_{x \to 0^+} g(x)$ exist? If so, what is its value? If not, why not?

(b) Does $\lim_{x \to 0^-} g(x)$ exist? If so, what is its value? If not, why not?

(c) Does $\lim_{x \to 0} g(x)$ exist? If so, what is its value? If not, why not?

5. Graph a function that meets all of the following criteria.

- $\lim_{x \to -\infty} h(x) = -\infty$
- $\lim_{x \to -1^-} h(x) = -3$
- $\lim_{x \to -1^+} h(x) = 2$
- $\lim_{x \to 3} h(x) = 4$
- $h(3) = 0$
- $\lim_{x \to \infty} h(x) = -1$

6. Each of the following statements are false. Find a counterexample to each of the statements. Your counterexample may take many different forms; i.e., you may come up with an equation, provide a graph, or give a detailed explanation.

   (a) For any function $f(x)$, $\lim_{x \to c} f(x) = f(c)$.

   (b) $\lim_{x \to \infty} \sin(x) = 0$ because $\frac{-1+1}{2} = 0$.

   (c) All limits of the form $\infty - \infty$ must go to zero.

   (d) If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} g(x) = L_2$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$.

7. Calculate the following limits using algebra.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^3+5x^2-14x}$

(b) $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$

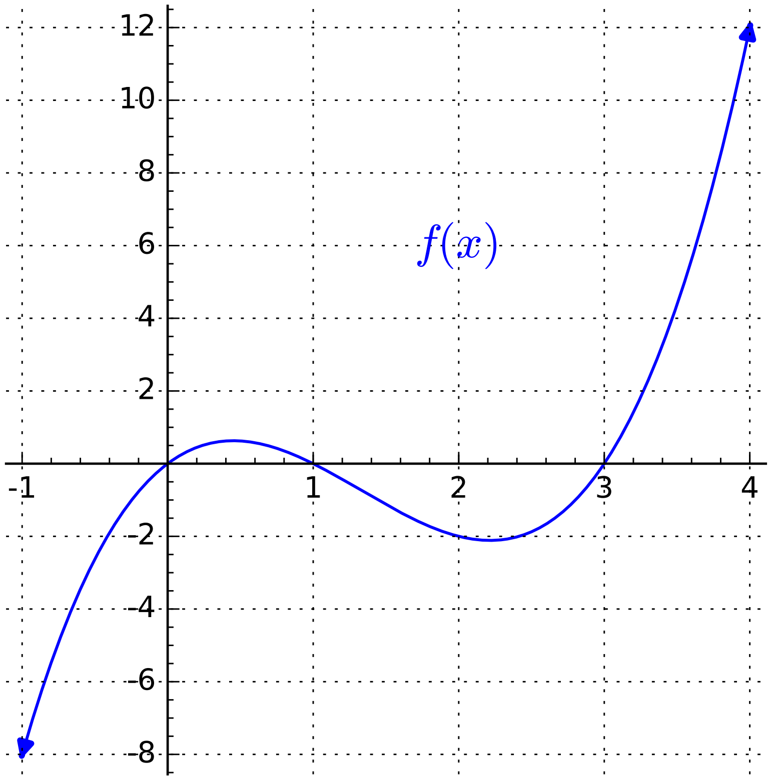
(c) $\lim\limits_{x\to \infty} \dfrac{x^7-11x^3-5x-2}{3x-2x^2-17x^{11}+12}$

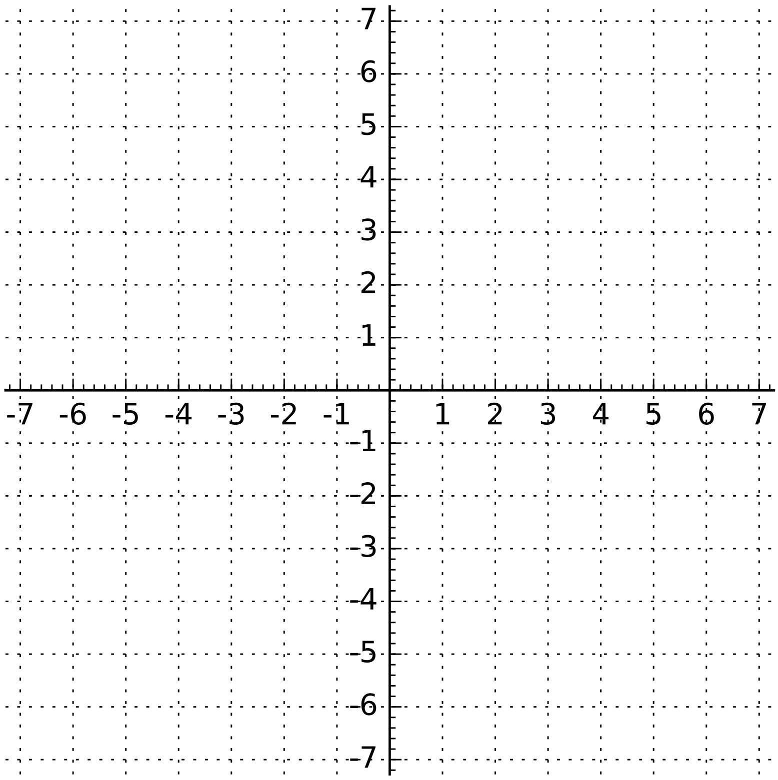
(d) $\lim\limits_{x\to 0} \dfrac{\sin(7x)}{2x}$

(e) $\lim\limits_{x\to 0} \dfrac{3e^x-3}{3e^{2x}+9e^x-12}$

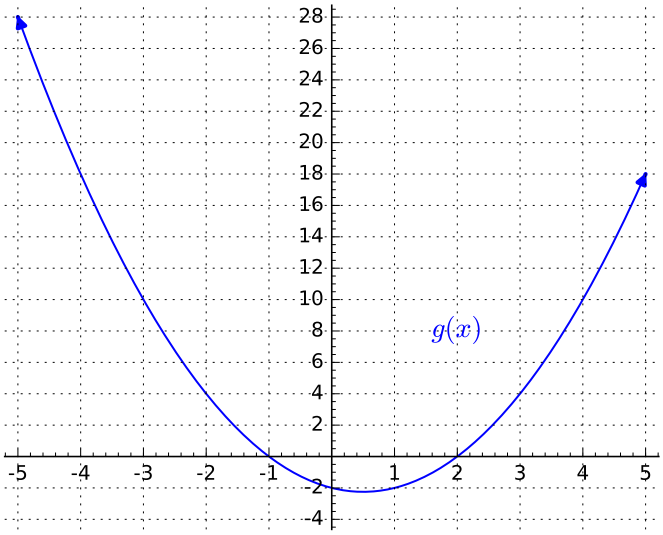
4 Section 2.1 – 2.3 Lab

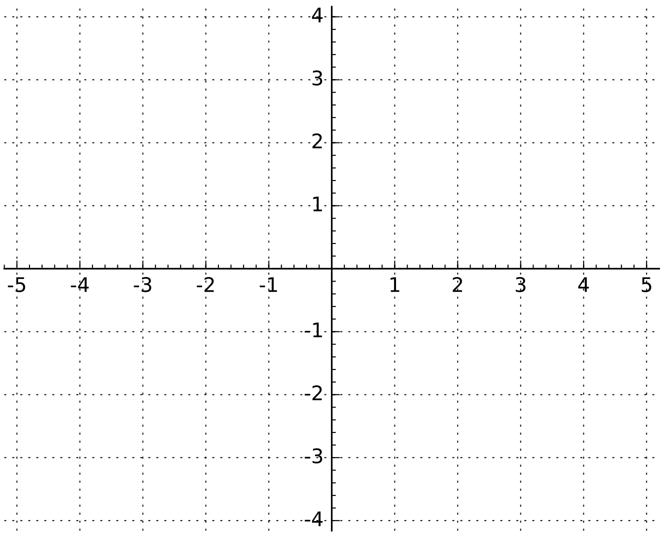
1. The graph of a function f is given. Sketch the graph of the associated slope function f'.





2. The graph of a function g is given. Sketch the graph of the associated slope function g'.

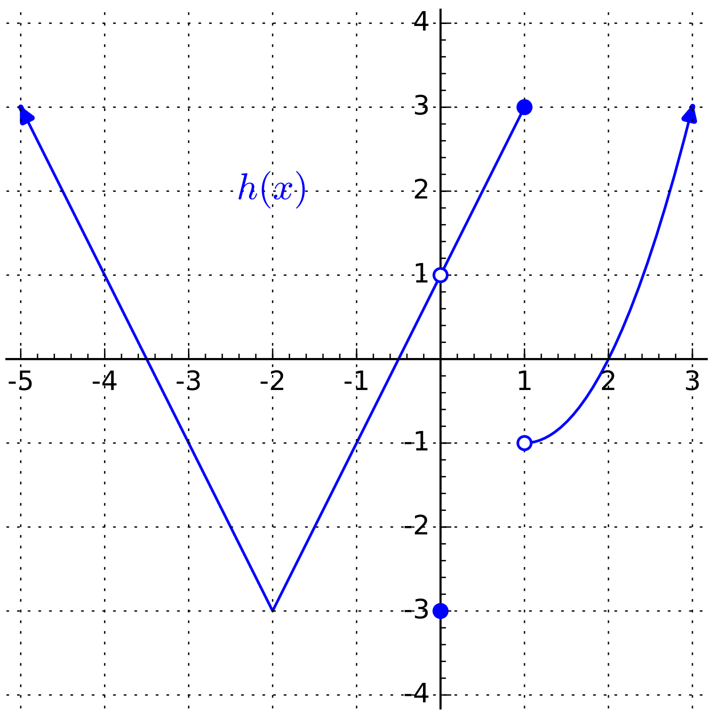




3. Use the limit definition of the derivative to find the derivative of $f(x) = \dfrac{2}{x+3}$.

4. Find the equation of the line tangent to the function $f(x) = 3x^2 + 4$ at $x = -1$.

5. For the function $h(x)$ graphed below, determine the values of x at which h fails to be continuous and/or differentiable. At each such point, determine the continuity (left-continuous and/or right-continuous) and the differentiablity (left-differentiable and/or right-differentiable). To indicate your answer, print the letter L if h is left-continuous (or left-differentiable) and/or the letter R if h is right-continuous (right-differentiable). Print the letter N if neither N nor R is appropriate. [There may be empty columns in the table.]



x-value					
continuity					
differentiability					

6. Use differentiation rules to find the derivative of the following functions or expressions. If a function (or expression) has domain restrictions, be sure to state those in your answer.

   (a) If $f(x) = 2x^3 + 4x - 3$, calculate $f'(x)$.

   (b) Calculate $\dfrac{d}{dx}\left(\dfrac{2x+5}{x^3}\right)$.

   (c) Calculate $\dfrac{d}{dx}\left(\dfrac{x^2-4}{x+2}\right)$.

   (d) If $g(x) = \sqrt{x}(2x^3 + x)$, calculate $g'(x)$.

5 Section 2.4 – 2.5 Lab

1. For each of the following, calculate the requested derivative.

   (a) $\dfrac{d}{dx}\left[\left(\dfrac{\sqrt{x}}{2}-1\right)^{-10}\right]$

   (b) $\dfrac{d}{dr}\left(\sqrt[3]{2r-r^3}\right)$

   (c) y', where $y=\dfrac{1}{2x^7+3x}$

(d) $\dfrac{d}{dx}\left(\ln\left(3x^2 - 4x + 1\right)\right)$

(e) $\dfrac{d}{d\theta}\left(\left(\theta^4 - \sqrt{3 - 4\theta}\right)^8 + 5\theta\right)$

(f) $\dot{x}$, where $x(t) = e^{t^2+3t} - \log(2t) + t^2$

2. Consider the implicitly-defined curve: $y^4 - 4y^2 = x^4 - 9x^2$.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the three (3) points on the implicit function where $x = -3$.

(c) Find the equation of the line tangent to the implicit curve at each of the points found in part (b).

3. If you drop a pebble into a large lake, you will cause a circle of ripples to expand outward. The area $A = A(t)$ and radius $r = r(t)$ of each circular ripple are both functions of t (indicating that they change over time) and are related by the formula $A = \pi r^2$.

   (a) If r is measured in inches and t is measured in seconds, what are the units of $\frac{dA}{dt}$? What are the units of $\frac{dA}{dr}$?

   (b) Find $\frac{dA}{dt}$ and explain in pratical terms the meaning of $\left.\frac{dA}{dt}\right|_{t=2}$.

6 Section 3.1 – 3.3 Lab

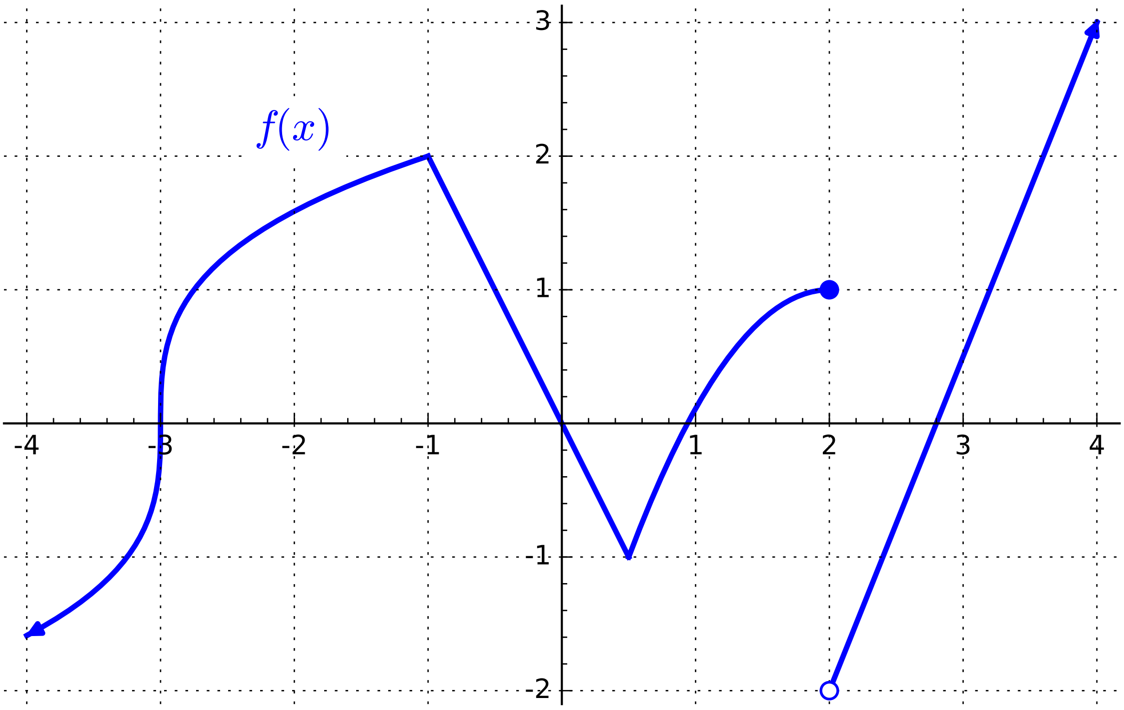
1. Determine whether the function $m(x) = \ln(x^2 - 4x + 3)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[4, 7]$. If it does, use derivatives and algebra to find the exact values of all $c \in (4, 7)$ that satisfy the conclusions of the Mean Value Theorem.

2. Explain why the following function does not satisfy the hypotheses of the Mean Value Theorem on its domain:

$$f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}.$$

3. A driver of an 18-wheel truck is traveling on a road with two toll stations. She pays for the toll at the first station, then two (2) hours and 161 miles later she arrives at the second toll. When she pays for the second toll she is given a ticket for speeding by the on-site state trooper. How did the trooper know that she was speeding?

4. Use the graph of $f(x)$ below to create the sign charts for f' and f''.

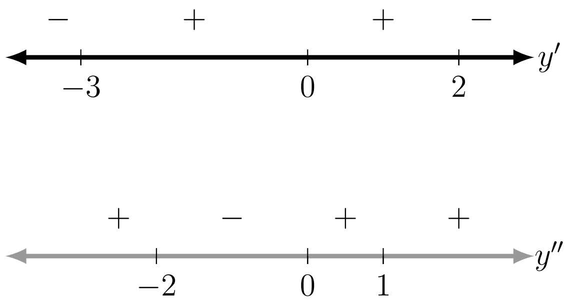


Sign Charts for f

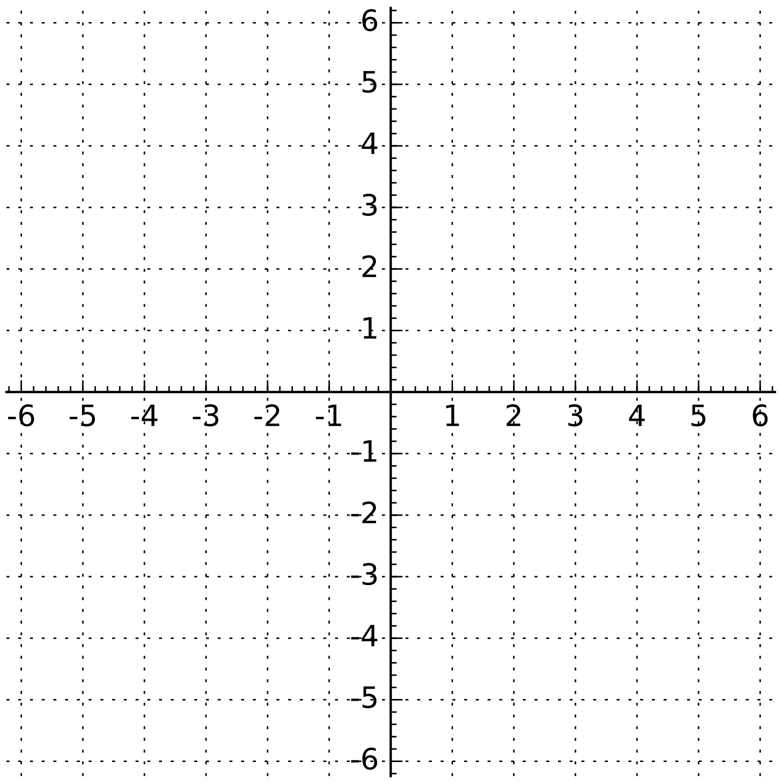
5. Let $g(x) = xe^x$. Create the sign charts for g' and g'' and use them to identify the local extrema.

Sign Charts for g

6. Below are the sign charts for a twice-differentiable function $y = f(x)$. Using the given sign charts and the fact that $f(x)$ passes through the point $(0, 1)$, sketch the graph of y.



Sign Charts for y



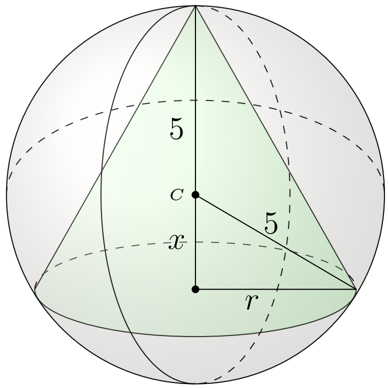
7. Consider the function $h(x)$ defined below. Determine all of the following characteristics of the function. Use ***exact*** values. NO DECIMAL ANSWERS!

$$h(x) = x^3 e^x$$

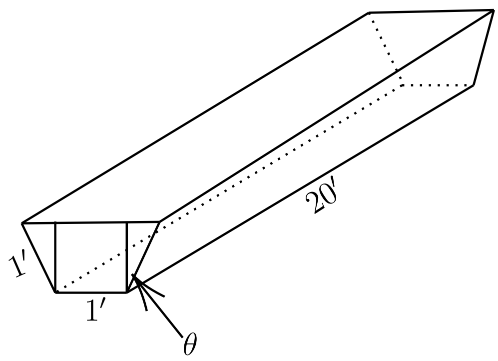
1. Critical point(s)/value(s)
2. Increasing interval(s)
3. Decreasing interval(s)
4. Inflection point(s)
5. Interval(s) where $h(x)$ is concave up
6. Interval(s) where $h(x)$ is concave down

7 Section 3.5 Lab

1. Find the volume of the largest right circular cone that can be inscribed inside a sphere of radius 5.



2. A 20-foot trough, whose bottom base is 1 foot wide, is made by creating an open trapezoidal prism. If the prism's volume is dependent upon the angle θ between the vertical height of the trapezoid and the side, then what value of θ would maximize the volume of the trough?



3. One interesting model that determines the concentration of caffeine in the blood based on nicotine consumption is given by the formula below:

$$c(t) = \frac{D}{1 - \frac{\beta}{\alpha}} \left(e^{-\beta t} - e^{-\alpha t} \right),$$

where D is the size of the dose of caffeine, α is the absorption rate of caffeine, and β is the elimination rate of caffeine. Assume that t is the only variable and all of the other letters represent constants. Find the time when the maximum concentration of caffeine is in the blood stream.

4. Two different masses, m_1 and m_2, hang from identical springs. The height of the first mass is given by $s_1 = 2\sin t$, and the height of the second mass is given by $s_2 = \sin 2t$.

   (a) Find the point(s) in the time interval $0 \le t \le 2\pi$ when the masses are at the same height. Give your answer in exact form.

   (b) Find the point(s) in the time interval $0 \le t \le 2\pi$ when the vertical distance between the masses is the greatest. Give your answer in exact form.

8 Section 4.1 – 4.2 Lab

1. Find $\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{k^2+k+1}{n^3}\right)$, if it exists.

2. Set up the sigma notation that calculates the signed area between the graph of $f(x) = 64 - x^2$ and the x-axis on the interval $[0, 8]$ using left sums for the following values of n. Then find each of these values using Desmos. [*Helpful information: To get the* Σ *symbol in Desmos, just type* `sum`.]

(a) When $n = 4$.

(b) When $n = 8$.

(c) Now, find the sum when $n \to \infty$. In other words, find the exact area.

3. Consider the function $f(x) = \dfrac{1}{x}$.

   (a) Write the right sums, in terms of n and k, that would approximate the area between $f(x)$ and the x-axis on the interval $[1, 5]$.

   (b) What is the smallest value of n, such that the difference between the approximate area and the value of $\ln(5)$ is less than 0.0001. This will require some guess-and-check using Desmos.

(b) ____________

4. Below is the graph of the function $g(x)$. Using geometry, calculate the exact area between the curve and the x-axis.

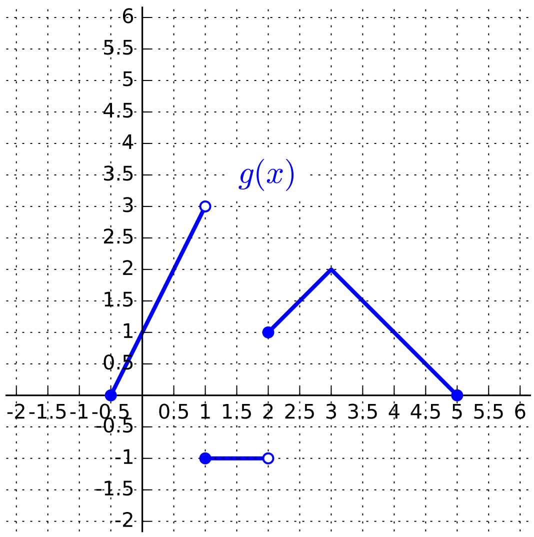


Figure 4: Graph of $g(x)$.

5. Construct a counterexample that illustrates why each of the following statements are false.

(a) If $\{a_n\}$ is a sequence of numbers, then $\sum_i (a_n)^2 = \left(\sum_i a_n\right)^2$.

(b) If $\{b_n\}$ is a sequence of numbers, then $\sum_i |b_n| = \left|\sum_i b_n\right|$.

(c) If $\{a_n\}$ and $\{b_n\}$ are both sequences of numbers, then $\sum_i (a_n \cdot b_n) = \left(\sum_i a_n\right)\left(\sum_i b_n\right)$.

9 Chapter 4 So Far

1. Using the summation formulas, determine the values of the following summations.

   (a) $\displaystyle\sum_{k=1}^{6} (2k+1)$

   (b) $\displaystyle\sum_{k=1}^{75} \left(2k^2\right)$

   (c) $\displaystyle\sum_{i=1}^{10} \left(3i^2 - 7i - 1\right)$

2. The following summations each represent the **right** Riemann sum of a given function for a finite number of rectangles. Identify the following pieces that make up the Riemann sum: $f(x)$, the interval $[a, b]$, n, and Δx.

(a) $\displaystyle\sum_{k=1}^{8} \left(e^{2+k\cdot\frac{1}{4}} \cdot \frac{1}{4} \right)$

- $f(x) =$ __________
- $[a, b] =$ __________
- $n =$ __________
- $\Delta x =$ __________

(b) $\displaystyle\sum_{k=1}^{100} \left(\tan\left(k \cdot \frac{\pi}{600} \right) \cdot \frac{\pi}{600} \right)$

- $f(x) =$ __________
- $[a, b] =$ __________
- $n =$ __________
- $\Delta x =$ __________

(c) $\displaystyle\sum_{k=1}^{4000} \left(\ln\left(3 + k \cdot \frac{e^2-3}{4000} \right) \cdot \frac{e^2-3}{4000} \right)$

- $f(x) =$ __________
- $[a, b] =$ __________
- $n =$ __________
- $\Delta x =$ __________

(d) $\displaystyle\sum_{k=1}^{16} \left(\left| -3 + k \cdot \frac{5}{8} \right| \cdot \frac{5}{8} \right)$

- $f(x) =$ __________
- $[a, b] =$ __________
- $n =$ __________
- $\Delta x =$ __________

3. Use geometry (i.e., areas of triangles, rectangles, and circles) to find the exact values of each of the following definite integrals.

(a) $\displaystyle\int_{-2}^{4} (|3x+1|)\, dx$

(b) $\displaystyle\int_{2}^{8} \left(2 + \sqrt{9-(x-5)^2} \right) dx$

4. Below is the graph of the function $h(x)$.

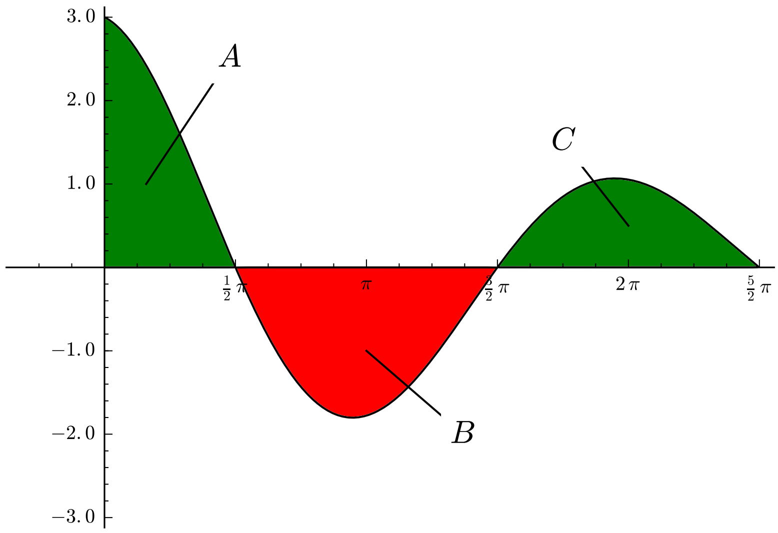


Figure 5: `https://goo.gl/ZbHfLR`

If the absolute areas of regions A, B, and C (rounded to three decimal places) are 2.733, 3.577, and 2.119, respectively, determine the values of the following definite integrals. Round your answers to three decimal places.

(a) $\int_0^{\frac{\pi}{2}} h(x)\,dx =$ ______

(b) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} h(x)\,dx =$ ______

(c) $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} h(x)\,dx =$ ______

(d) $\int_0^{\frac{5\pi}{2}} h(x)\,dx =$ ______

(e) $\int_0^{\frac{3\pi}{2}} 2h(x)\,dx =$ ______

(f) $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} |h(x)|\,dx =$ ______

(g) $\int_0^{\frac{\pi}{2}} [h(x)+2]\,dx =$ ______

(h) $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} [1-h(x)]\,dx =$ ______

(i) $\int_{\frac{5\pi}{2}}^{0} 2|h(x)|\,dx =$ ______

5. The **floor** function is defined to the *the greatest integer less than or equal to* x and is denoted by the symbol $\lfloor x \rfloor$. Using geometry, find the value of the following definite integral:

$$\int_1^5 x\lfloor x\rfloor\,dx.$$

[The following Desmos graph may be useful: `https://www.desmos.com/calculator/l5cw5x57uc`.]

6. Use integration formulas to solve each of the following integrals. You may have to use algebra, educated guess-and-check, and/or recognize an integrand as the result of a product, quotient, or chain rule calculation. You can check each of your answers by differentiating.

(a) $\displaystyle\int \left(x^3+4\right)^2\,dx$

(b) $\displaystyle\int \frac{3}{1+x^2}\,dx$

(c) $\displaystyle\int 2\sin(x)\cos(x)\,dx$

(d) $\displaystyle\int 3x^2\cos(x^3+1)\,dx$

(e) $\displaystyle\int \left(2x\sin(x) + x^2\cos(x)\right)\, dx$

(f) $\displaystyle\int \left(\frac{-\sin(x)\ln(x) - \frac{\cos(x)}{x}}{(\ln(x))^2}\right)\, dx$

(g) $\displaystyle\int \left((2x+3)e^{x^2+3x+2} + \cos(x)\right)\, dx$

(h) $\displaystyle\int |\sin(x)|\cot(x)\, dx$ $\left[\text{Hint: Change } \cot(x) = \frac{\cos(x)}{\sin(x)}\right]$

7. Use the Fundamental Theorem of Calculus to find the exact values of each of the definite integrals given below. You are encouraged to use Desmos to check your answer, but only exact answers will be accepted.

(a) $\displaystyle\int_0^1 \left(\frac{1}{2e^x}\right)\, dx$

(b) $\displaystyle\int_5^{11} \left(\frac{1}{x-3}\right)\, dx$

(c) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sec^2(x)\right)\, dx$

(d) $\displaystyle\int_{-1}^{1} \left(\frac{e^x - xe^x}{e^{2x}}\right)\, dx$