I think there are two definition about convex, one for functions and one for sets. The first question is talking about "functions" are convex; while the second and the third questions are talking about sets.

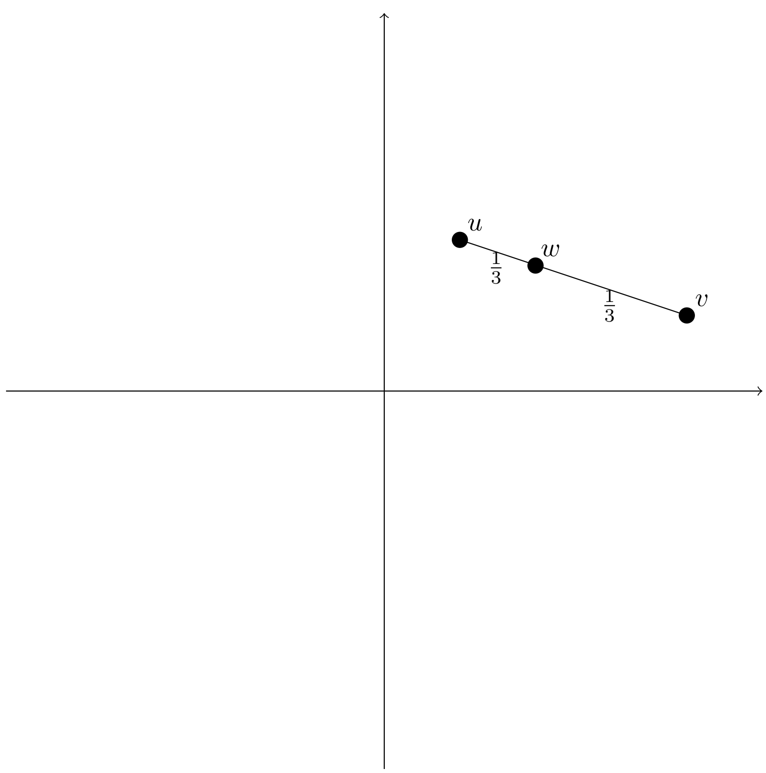
Before we start, here is some background. Suppose u and v are two points in a space (this includes lines for 1d, planes for 2d, and spaces for 3d). Let $0 \leq t \leq 1$. The formula $w = tu + (1-t)v$ is a point w on the line between u and v, with the property that

$$|w - u| : |w - v| = 1 - t : t.$$

For example, $u = (1, 2)$, $v = (4, 1)$, and $t = \frac{2}{3}$. Then

$$w = tu + (1-t)v = (\frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4, \frac{2}{3} \cdot 2 + \frac{1}{3} \cdot 1) = (2, \frac{5}{3}).$$

In particular, when $t = \frac{1}{2}$, w is the middle point.



Question 1 Starting with the first question. A function f is called *convex* if it has the property

$$f(w) \leq tf(u) + (1-t)f(v),$$

for all u, v, t, and $w = tu + (1-t)v$. This basically means the function is having a mouth facing up. For example, $f(x) = x^2$ is a parabola, which is convex; in contrast, $f(x) = -x^2$ has the mouth facing down, so it is not convex. Also, a function like $f(x) = \sin x$ is oscilating, so it is not convex either.

For the first question, you need to prove that

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$$

for all possible u and v. To do this, you may use the fact that $f_1, f_2, \ldots, f_m$ are all convex. Example of the proof is provided below.

Proof. Let $u, v \in \mathbb{R}^n$. Then

$$f_i(tu + (1-t)v) \leq tf_i(u) + (1-t)f_i(v)$$

for all i, since f_i's are convex. Also, taking the maximum preserve the inequality, so ... □

Question 2 and 3 A set X is called *convex* if for any two point $u, v \in X$, $tu + (1-t)v$, $0 \leq t \leq 1$, is also a point in X. This is a formal definition, but intuitively what it mean is as follow: for any two points $x, y \in X$, the segment connecting u and v is also lying in X. For example, a disc is convex; however, a circle is not, since circle does not contain the inner part.