CIRCUMBILLIARD GEOMETRY

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Abstract. Mathematical billiards are at crossroads of various research works, but mainly covered in dynamicals systems. Dan Reznik found geometric invariants for the inner elliptic billiard. We propose a parametrization of a 3-orbit, a "circumbilliard", using euclidean plane geometry.

1. Introduction

In this note we present analytical and geometrical results about 3-orbits in an elliptic billiard. They are deduced from Dan Reznik [1] observations in the flavor of plane euclidean geometry. In physical or optical terms, we restrict the results to "specular reflection" on the ellipse, a perfect elastic reflection inducing a Snell law.

2. Definition in triangle geometry

We define a *circumbilliard* as a not oriented 3-orbit ABC in an elliptical billiard.

Points A,B,C are reflections points of a modelized blue "ball" on the elliptic boundary.

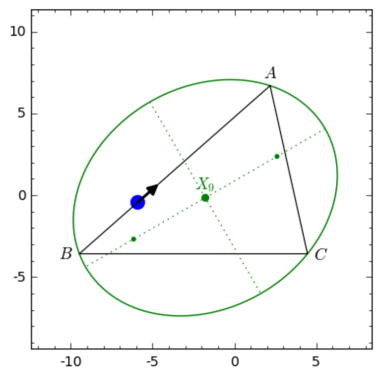

Figure 1. Circumbilliard for an ellipse.

The 3-orbit ABC is choosen as the *reference triangle* in triangle geometry. Common notation is used for triangle vertices (A,B,C) and triangle edge lengths ($a = BC$, $b = CA$, $c = AB$).

The ellipse is named *circumbilliard* by analogy with the ABC *circumcircle,* the unique circle going through A,B and C.

Point of view is changed versus representing orbits in a fixed ellipse : here we set a triangle ABC, and draw the ellipse centered at Mittenpunkt X_9.

3. Parametrization

One invariant for n-orbits in elliptic billiards is total length of the orbit, an extremal value (*Lemma 2.3* in [2]) .For a 3-orbit ABC it is twice the semi-perimeter s.

Another invariant discovered by Dan Reznik is the ratio between inradius and circumradius, or equivalently the sum of cosines at internal angles at A,B and C vertices (*Theorem 1* in [2]))

We use these invariants to find ABC edges lengths parametrization.

Theorem 3.1. *Edges lengths of a 3-orbits ABC can be defined as :*

$$a = \frac{2(1-t)s}{k-2t+2}$$
$$b = \frac{(kt-t^2+k+1-w)s}{(1+t)(k-2t+2)}$$
$$c = \frac{(kt-t^2+k+1+w)s}{(1+t)(k-2t+2)}$$

with

$$w = \pm\sqrt{(1-t^2)(h^2-(t-k)^2)}$$
$$h = \sqrt{1-2k}$$

and parameter t in [$k-h$,$k+h$] is cosine internal angle A.

Proof. We use Ravi substitution : $a = y+z$, $b = z+x$, $c = x+y$ to compute some geometric values for triangle ABC. The next formulas are from triangle geometry.

Formula for semi-perimeter :

$$s = \frac{1}{2}(a+b+c) = x+y+z$$

Formula for circumradius :

$$R = \frac{abc}{4S} = \frac{(x+y)(x+z)(y+z)}{4S}$$

where S is ABC area.

Inradius :

$$r = \frac{S}{s} = \frac{2S}{x+y+z}$$

Ratio between inradius and circumradius :

$$k = \frac{r}{R} = \frac{4xyz}{(x+y)(x+z)(y+z)} = k$$

Cosines of internal angles using cosines law :

$$\cos(A) = \frac{b^2+c^2-a^2}{2bc} = \frac{x^2+x(y+z)-yz}{(x+y)(x+z)} = t$$
$$\cos(B) = \frac{c^2+a^2-b^2}{2ca} = \frac{y^2+y(z+x)-zx}{(y+z)(y+x)}$$
$$\cos(C) = \frac{a^2+b^2-c^2}{2ab} = \frac{z^2+z(x+y)-xy}{(z+x)(z+y)}$$

Adding all cosines and factoring :

$$\cos(A) + cos(B) + cos(C) = \frac{x^2(y+z) + y^2(z+x) + z^2(x+y) + 6xyz}{(x+y)(x+z)(y+z)}$$

or

$$\cos(A) + cos(B) + cos(C) = 1 + \frac{4xyz}{(x+y)(x+z)(y+z)} = 1 + k$$

From previous formulas we deduce variables x,y,z are roots of :

(1) $x + y + z - s = 0$

(2) $k(x+y)(x+z)(y+z) - 4xyz = 0$

(3) $x^2 + xy + xz - yz - t(x+y)(x+z) = 0$

We choose the couple of first and third equations, expressing y and z from x,k,s and t. Putting back expression into second equation we get :

$$x = \frac{ks}{k - 2t + 2}$$

The first and third equations give sum and product of y and z :

$$\sigma = y + z = s - x = \frac{2(1-t)s}{k - 2t + 2}$$

$$\mu = yz = \frac{1 - t)x(x + y + z)}{1 + t} = \frac{(1-t)ks^2}{(1+t)(k - 2t + 2)}$$

Hence y,z are roots of the binomial equation : $U^2 - \sigma U + \mu = 0$.

$$y = \frac{(1 - t^2 + w)s}{(1+t)(k - 2t + 2)}$$

$$z = \frac{(1 - t^2 - w)s}{(1+t)(k - 2t + 2)}$$

with

$$w^2 = (1 - t^2)(1 - 2k - (t - k)^2)$$

Substituting x,y,z into a,b,c we get a,b,c parametrization in k,s,t.

□

The common part in a,b,c parametrization ($\frac{s}{k-2t+2}$) is a scaling factor very useful in trilinear coordinates computations. Changing sign of w is simply swapping b and c or equivalenty reflecting the 3-orbit on minor axis.

4. Equation in trilinear coordinates

Something is missing in the definition of the *circumbilliard ellipse* because we need five points, no three aligned, to define an ellipse and here we have only three of them : A,B,C.

Dan Reznik found two more points which are on the ellipe for all 3-orbits, points X_{88} (isogonal conjugate of X_{44}) and X_{100} (Feuerbach anticomplement point):

$$X_{88} = \frac{1}{b+c-2a} : \frac{1}{c+a-2b} : \frac{1}{a+b-2c}$$
$$X_{100} = \frac{1}{b-c} : \frac{1}{c-a} : \frac{1}{a-b}$$

Property for these points : the sum of denominators of trilinear coordinates is 0.

These points are sometimes not well defined (for example, for isosceles ABC), that's why, we prefer the reflection of B and C with respect to Mittenpunkt X_9 center of the ellipse in the proof of the next theorem.

Theorem 4.1. *Circumbilliard ellipse* (Φ) *is locus of points* $M = \alpha : \beta : \gamma$ *whose trilinear coordinates are satisfying equation :*

$$\alpha\beta + \alpha\gamma + \beta\gamma = 0$$

which is the isogonal conjugate of the line (Φ^*) *with equation*

$$\alpha + \beta + \gamma = 0$$

Proof. Computing trilinear coordinates of B, and C reflections with respect to Mittenpunkt we get :

$$D = 2(-a+b+c)b : (a+b-c)(a-b-c) : 2(a+b-c)b$$
$$E = 2(-a+b+c)c : 2(a-b+c)c : (a-b+c)(a-b-c)$$

Isogonal conjugates of these points are :

$$D^* = \frac{1}{(-a+b+c)b} : \frac{2}{(a+b-c)(a-b-c)} : \frac{1}{(a+b-c)b}$$
$$E^* = \frac{1}{(-a+b+c)c} : \frac{1}{(a-b+c)c} : \frac{2}{(a-b+c)(a-b-c)}$$

Trilinear coordinates $\alpha : \beta : \gamma$ of D^* and E^* are satisfying the line equation $\alpha + \beta + \gamma = 0$. Moreover these two points are distinct when the ellipse is not a circle ($k < \frac{1}{2}$) because the squared distance is not 0 :

$$(D^*E^*)^2 = (1-2k)\left(\frac{2(k+4)(1-t)s}{4k^2t - 6kt^2 + 5k^2 - 4kt + 12t^2 + 10k - 8t - 4}\right)^2$$

The ABC circumcircle has equation :

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$$

The isogonal conjugate of a line is an ellipse through A,B,C if and only if the line doesn't intersect the ABC circumcircle ([3]).

Substituting α with $-\beta-\gamma$, the circumcircle intersects the line D^*E^* with equation $\alpha+\beta+\gamma=0$ if only if it exists one real root β of equation :

$$c\beta^2+(-a+b+c)\beta\gamma+b\gamma^2=0$$

The squared discriminant of this binominal equation is γ^2 times Δ :

$$\Delta=a^2-2ab+b^2-2ac-2bc+c^2=\frac{-4k(k+4)(1-t)s^2}{(k-2t+2)^2(1+t)}$$

is always negative, and there is no real root.

The isogonal conjugate of the line D^*E^* is the circumbilliard ellipse with equation : $\alpha\beta+\alpha\gamma+\beta\gamma=0$.

□

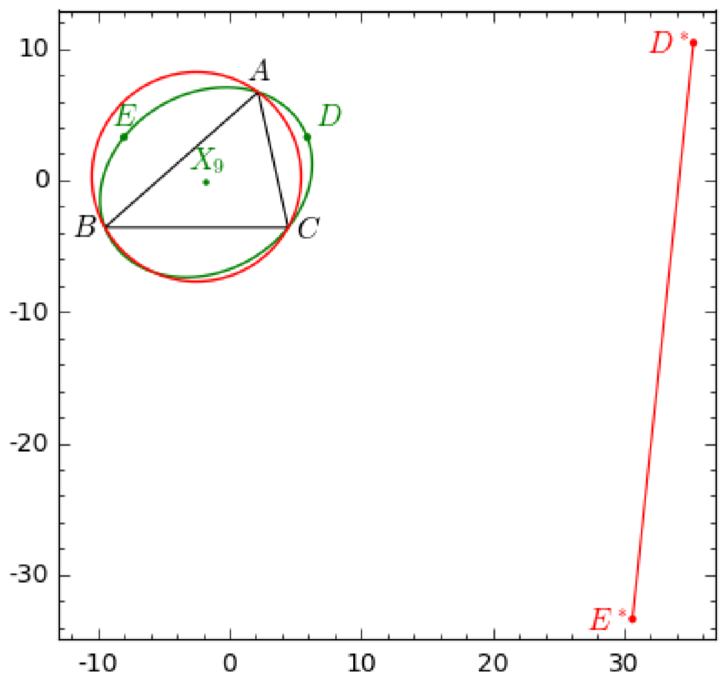

FIGURE 2. Isogonal conjugation

We can choose many couple of points (P^*,Q^*) instead of (D^*,E^*) to define the line (Φ^*).

A simple choice is $P^*=1:-2:1$ and $Q^*=1:1:-2$ with :

$$(P^*Q^*)^2=\frac{36(1-2k)k^2(1-t)^2s^2}{(4kt-2t^2-5k+2)^2(k-2t+2)^2}$$

The line (Φ^*) is the *center line* $X_{44}X_{513}$ and *triangle centers* on that line have isogonal conjugates on the ellipse (Φ) : triangle centers X_{88}, X_{100}, X_{190}, X_{651}, X_{162}, X_{660}, X_{662}, X_{673}, X_{799}, X_{823}, X_{897}, X_{31002} and so on.

5. Ellipse axis

It is easy to compute the two intersection points of a line : $l = l_a : l_b : l_c$ equation $l_a\alpha + l_b\beta + l_c\gamma = 0$ with the ellipse equation $\alpha\beta + \alpha\gamma + \beta\gamma = 0$.

We set a generic point $P = 0 : (1-m)c : mb$ on the BC line and intersect line X_9P with the ellipse to get two points M_i and M_j.

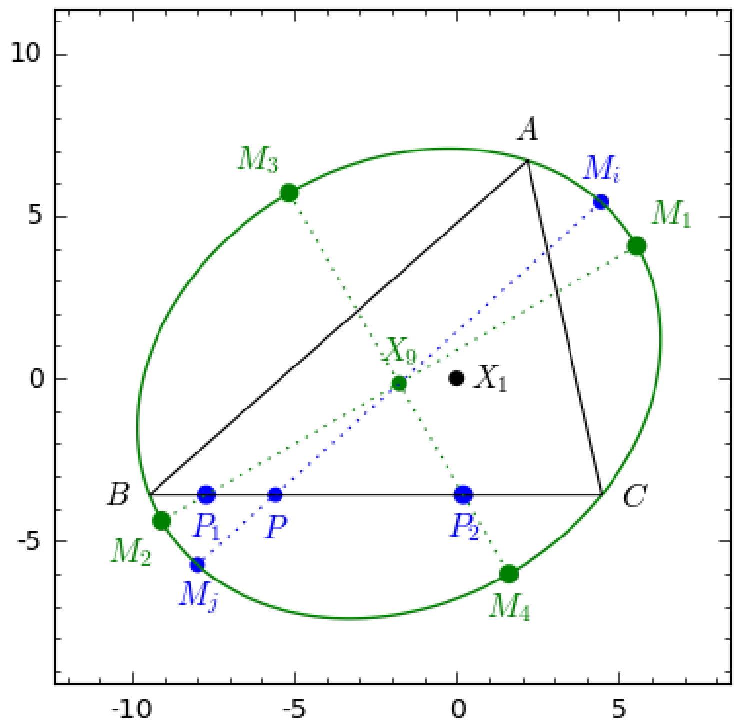

Figure 3. Axis

We maximize or minimize function $q_{ij}(m) = (M_iM_j)^2$ the squared distance between M_i and M_j to get the optimal values for m : $m = m_1$ for major axis M_1M_2 and $m = m_2$ for minor axis M_3M_4.

$$m = \frac{-c(a^2+b^2+c^2+ab-2ac-2bc) \pm \Delta}{(a^2+b^2+c^2-2ab-2ac-2bc)(b-c)}$$

with

$$\Delta^2 = (a^3+b^3+c^3-a^2b-ab^2-a^2c-b^2c-ac^2-bc^2+3abc)abc$$

.

Using parametrization we deduce formulas for squared major and minor lengths :

$$p^2 = (X_9M_1)^2 = \frac{4(1+h)s^2}{(3-h)(3+h)^2}$$

$$q^2 = (X_9M_3)^2 = \frac{4(1-h)s^2}{(3+h)(3-h)^2}$$

and focal distance : $f = \frac{4\sqrt{h}s}{9-h^2}$.

6. Cartesian coordinates

Drawings have been done previously with the following layout : BC segment is set horizontal, A is drawn up BC and incenter X_1 is set at cartesian plane origin point with (0,0) as cartesian coordinates.

It is convenient for dynamics in elliptic billiard to choose another layout where the center of ellipse X_9 is at (0,0) and major, minor axis are horizontal and vertical. The 3-orbits are then drawn as triangles ABC inscribed in the fixed ellipse.

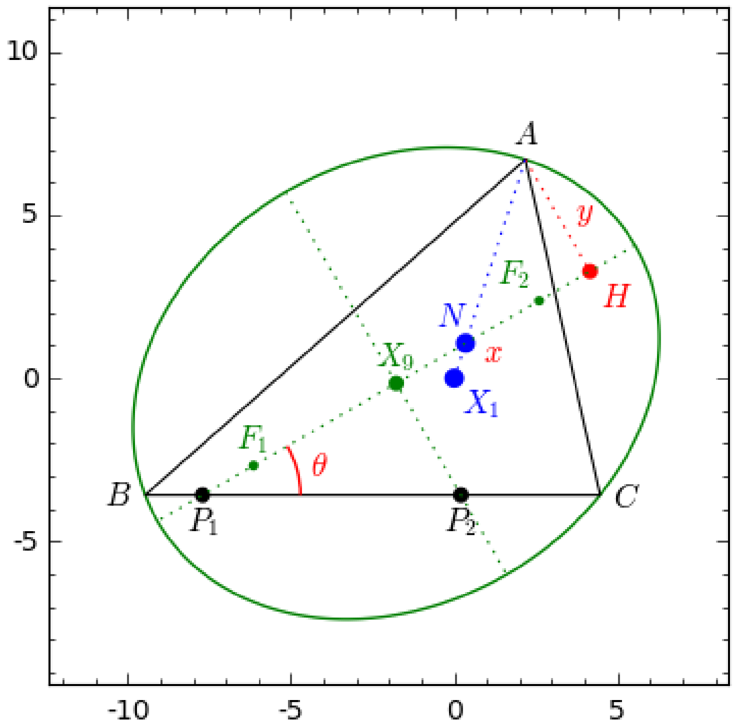

Figure 4. Triangle geometry layout

We will morph drawing using a translation and a rotation.

Translation vector from $(X_1, X_9) = (x_t, y_t)$ with :

$$x_t = \frac{2(h^2t - 5t + 4)s}{(h^2 + 4t - 5)(9 - h^2)} \sqrt{\frac{4h^2 - (h^2 + 2t - 1)^2}{1 - t^2}}$$

$$y_t = \frac{2(h^2t + 2t^2 - 5t + 2)(1 - h^2)s}{(h^2 + 4t - 5)(9 - h^2)} \sqrt{\frac{1}{1 - t^2}}$$

Cosine and sine of rotation angle $\theta_t = \theta$ are computed using edge lengths triangle P_1X$_9$P$_2$:

$$\cos\theta_t = \frac{1 + h}{2} \sqrt{\frac{2h - h^2 - 2t + 1}{2h(1 - t)}}$$

$$\sin\theta_t = \frac{1 - h}{2} \sqrt{\frac{2h + h^2 + 2t - 1}{2h(1 - t)}}$$

Ellipse equation in cartesian coordinates is given from p^2 and q^2 formulas :

$$(3+h)(1-h)x^2 + (3-h)(1+h)y^2 = \frac{f^2(1-h^2)(9-h^2)}{4h} \tag{1}$$

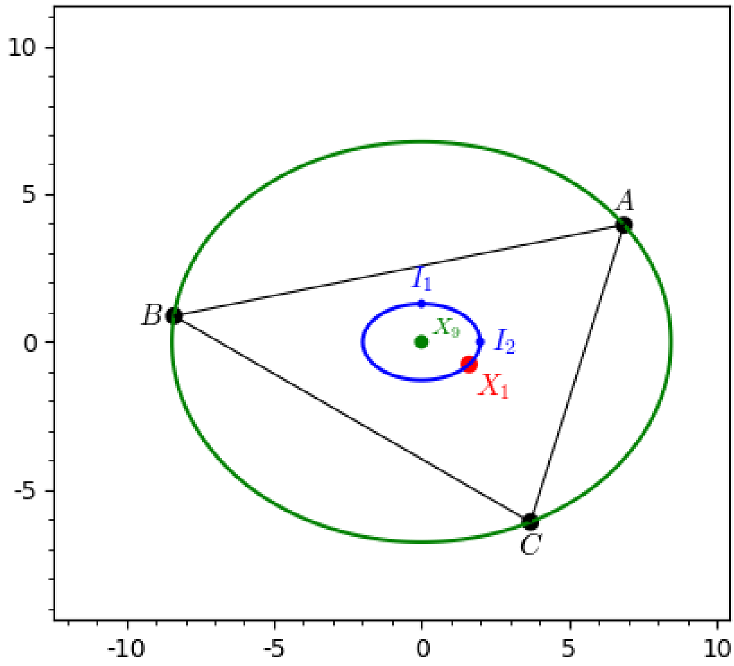

FIGURE 5. Cartesian coordinates layout

Using ellipse equation (1) we can deduce the squared cartesian coordinates x^2 and y^2 for any point M on the ellipse such as $(X_9M)^2 = x^2 + y^2$. For A,B,C vertices we get :

$$x_A^2 = \frac{(h^2+2h+2t-1)(1+h)^2 s^2}{2(3+h)^2 h(1+t)}$$

$$y_A^2 = \frac{(-h^2+2h-2t+1)(1-h)^2 s^2}{2(3-h)^2 h(1+t)}$$

$$x_B^2 = \frac{(u+v-2(h^2+2h-3)w)(1+h)^2 s^2}{2(h^2+4t-5)^2(3+h)^2 h}$$

$$y_B^2 = \frac{(-u+v+2(h^2-2h-3)w)(1-h)^2 s^2}{2(h^2+4t-5)^2(3-h)^2 h}$$

$$x_C^2 = \frac{(u+v+2(h^2+2h-3)w)(1+h)^2 s^2}{2(h^2+4t-5)^2(3+h)^2 h}$$

$$y_C^2 = \frac{(-u+v-2(h^2-2h-3)w)(1-h)^2 s^2}{2(h^2+4t-5)^2(3-h)^2 h}$$

with

$$u = 2h^4 t + 4h^2 t^2 - 8h^2 t + 20t^2 - 26t + 8$$
$$v = 4(h^2+2t-7)h(t-1)$$

7. Triangle centers

We use homogenous barycentric coordinates to get parametrization of cartesian coordinates of a triangle center. Homogenous barycentric coordinates for $M = \alpha : \beta : \gamma$ are $(\frac{a\alpha}{n}, \frac{b\beta}{n}, \frac{c\gamma}{n})$ with $n = a\alpha + b\beta + c\gamma$.

As an example we compute cartesian coordinates (x_I, y_I) of ABC incenter $I = X_1$ whose locus is drawn as the blue ellipse figure 5.

The subject was dealt by Olga Paris-Romaskevich in [4] teasing for use of *complex reflection*. Ronaldo Garcia added explicit cartesian coordinates equation of the locus in [5].

Incenter has constant coordinates : $\alpha = \beta = \gamma = 1$. From them we get barycentric coordinates with respect to A,B,C points :

$$k_A = \frac{1+h^2}{3+h^2}$$

$$k_B = \frac{3-h^2-h\sqrt{-h^2+2h^2+3}}{9-h^4}$$

$$k_C = \frac{3-h^2+h\sqrt{-h^2+2h^2+3}}{9-h^4}$$

and

$$x_I = k_A x_A + k_B x_B + k_C x_C$$

$$y_I = k_A y_A + k_B y_B + k_C y_C$$

Usually these cartesian coordinates don't have simple closed form. Signs of A,B,C cartesian coordinates are troublesome too because depending on t interval values range.

If we already know that geometric locus is a conic, it is easier. For an ellipse, we plug into cartesian coordinates formulas five values of t between $k-h$ and $k+h$ to get five points and then deduce ellipse equation.

$$\frac{x^2}{l^2} + \frac{y^2}{m^2} = 1$$

Cheating here a little because I_1 (for $t = k-h$) and I_2 (for $t = k+h$) are on minor and major axis we get easily :

$$l^2 = \frac{16(1+h)h^2s^2}{(3+h)^2(3-h)^3}$$

$$m^2 = \frac{16(1-h)h^2s^2}{(3+h)^3(3-h)^2}$$

from plugging $t = k \pm h$ in $x_A^2(t)$, $y_A^2(t)$, $x_B^2(t)$, $y_B^2(t)$, $x_C^2(t)$, $y_C^2(t)$, square rooting and evaluating $x_I(t)$ and $y_I(t)$.

From l^2 and m^2, we compute local distance : $4s(\sqrt{\frac{2h}{9-h^2}})^3$ which is one of result of [5].

8. Other formulas

From ellipse equation and squared distance formula

$$(X_9A)^2 = \frac{-(a^4+b^4+c^4-2(a^2+2bc)(b^2+c^2)+6b^2c^2)bc}{(a^2+b^2+c^2-2(ab+bc+ca))^2}$$
$$= \frac{-2(h^4+4h^2t-6h^2-12t-3)s^2}{(h+3)^2(h-3)^2(1+t)} = q$$

we can deduce parameter t :

$$t = -\frac{2(h^4-6h^2-3)s^2+(h+3)^2(h-3)^2q}{8(h^2-3)s^2+(h+3)^2(h-3)^2q}$$

Squared distance from X_9 to B and C can be computed by permuting a,b,c cyclically in $(X_9A)^2$ formula or replacing $t = \cos A$ with :

$$\cos B = \frac{4(t+1)(t-1)^2-(h^2+2t-3)w}{2(t-1)((h^2+2t-3)(t+1)-w)}$$

or

$$\cos C = \frac{4(t+1)(t-1)^2+(h^2+2t-3)w}{2(t-1)((h^2+2t-3)(t+1)+w)}$$

9. Future work

We will compute radius of curvature, intersection point of A angle bissector with major axis, generalize the Ptolemy-Alhazen circle inner reflection problem to an ellipse and deal with n-orbits.

References

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