Triangle cyclic hinged dissection with 120-degrees triangles

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March 6, 2019

Abstract

We show that the geometric dissection of a triangle into five triangles with 120 degrees internal angle exists for a family of triangles.

1 Introduction

Focus is given here to the geometrical dissection of plane polygonal shape starting with the simplest one : a triangular shape (lamina in real world).

Practical usage of that dissection is in computational geometry, for morphing hinged cyclic polygonal chain with slender adornments.

There are many ways to dissect a triangle, but we want a dissection into five triangles with obtuse 120-degrees internal angle.

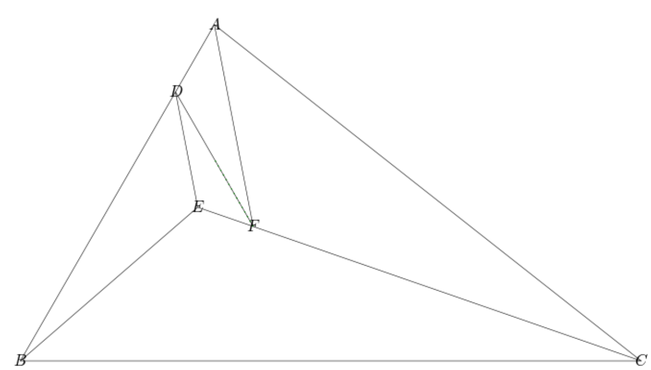

Figure 1: Model for dissection is perfect 8:7:5 triangle

The cyclic hinged dissection is given by the polygonal chain B-C-A-F-D-B-E-B where all the points from A to F are hinges, and the five 120 degrees triangles are *slender adornments* of the chain.

2 Main result

Dissections of a triangle into similar triangles have been dealt by [1] Zak. When all triangles are not congruent pairwise then it is a *perfect dissection*.

The pictured 8:7:5 triangle dissection is model pattern for a perfect dissection of a triangle into five triangles. Bonus property : they are 120 degrees triangles.

Theorem 1 (Perfect triangle model dissection)**.** *Not too much A-sharp triangles can be five 120-degrees triangles cyclic hinged dissected.*

Geometrical proof :

We will call ABC the *reference triangle* because we will do algebraic computation in trilinear coordinates after the geometric explanation.

Given an ABC triangle, we first set a point D on the AB edge, choosing any point on the line segment between A and B.

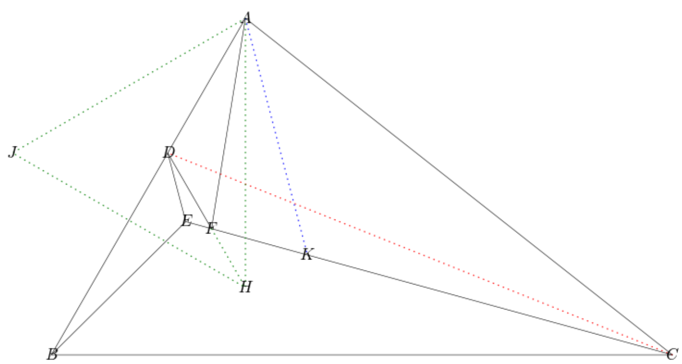

Figure 2: Geometric dissection

Construct E the first Fermat point of the BCD triangle. It has the following property that BEC, CED and DEB are obtuse triangles with an internal 120 degrees angle at E vertex.

It is easily done by intersecting two Simpson lines : set points C_1 and D_1 defining equilateral DC_1B and BD_1C triangles outside ABC. Intersect CC_1 with DD_1 to get E point.

Build an equilateral triangle AJH such as D is its centroid and AD is lying on the altitude line from A. Intersect DH with CE to get F point. Internal angle at D for ADH and ADF is 120 degrees.

Intersect CE with parallel of DE thru A to get K. Because parallel segments DE and AK on CE, angles DEC and AKC are equal, with 120 degrees value. Another way : intersect CE with circumcircle equilateral triangle ACL.

Move D until $F = K$ to find the dissection. Lines AF,DE are parallel.

This construction works for ABC triangles with angle A *not too much sharp* as we show in the algebraic proof.

Algebraic proof :

We use the usual notations for trilinear computations with a,b,c the respective lengths of edges BC,CA,AB of the reference triangle ABC, S for area of ABC, and $q_a = a^2$, $q_b = b^2$, $q_c = c^2$ for the quadrances.

D is cutting AB segment in x ratio and finding a dissection is equivalent to find x value such as AF,DE are parallel.

Trilinear coordinates of D are : $D = [\frac{1-x}{a}, \frac{x}{b}, 0]$.

Using Conway formula for an equilateral triangle BD_1C build on edge BC :

$$D_1 = [-6abc, c(4\sqrt{3}S + 3(a^2 + b^2 - c^2)), b(4\sqrt{3}S + 3(a^2 - b^2 + c^2))]$$

Using Conway formula for an equilateral triangle BC_1D build on edge BD :

$$C_1 = [b(3(a^2 - b^2 + c^2) + 4\sqrt{3}S), a(3(-a^2 + b^2 + c^2) + 4\sqrt{3}S\frac{1+x}{1-x}), -6abc]$$

Intersecting $CC1$ and $DD1$ we get trilinear coordinates for E point :

$$E = [-bc(3(a^2-b^2+c^2)x+3(a^2+b^2-c^2)+4\sqrt{3}S(1-x))(3(a^2-b^2+c^2)+4\sqrt{3}S)(x-1),$$
$$ac(3(a^2-b^2+c^2)x+3(a^2+b^2-c^2)+4\sqrt{3}S(1-x))(3(a^2-b^2-c^2)x+3(-a^2+b^2+c^2)+4\sqrt{3}S(1+x)),$$
$$ab(6c^2x + 3(a^2 - b^2 - c^2) - 4\sqrt{3}S)(3(a^2 - b^2 + c^2) + 4\sqrt{3}S)(x - 1)]$$

Using Conway formula for a 120 degrees triangle AHD build on segment AD :

$$H = [b(4(-3 + \frac{2}{x})S - \sqrt{3}(a^2 - b^2 + c^2)), a(12S + \sqrt{3}(a^2 - b^2 - c^2)), 2\sqrt{3}abc]$$

Intersecting DH and CE we get trilinear coordinates for F point :

$$F = [-b(6c^2x - 3(-a^2 + b^2 + c^2) + 4\sqrt{3}S)(3(a^2 - b^2 + c^2) + 4\sqrt{3}S),$$
$$a(6c^2x - 3(-a^2 + b^2 + c^2) + 4\sqrt{3}S)(3(a^2 - b^2 - c^2) + 4\sqrt{3}S\frac{1+x}{1-x}),$$
$$6abc(6c^2x - 3(-a^2 + b^2 + c^2) - 4\sqrt{3}S)]$$

Intersecting CE with a parallel line of DE thru A we get K point :

$$K = [bc(8\sqrt{3}Sx + 3(-a^2 - b^2 + c^2) - 4\sqrt{3}S)(3(a^2 - b^2 + c^2) + 4\sqrt{3}S)(x - 1),$$
$$-ac(3(a^2-b^2-c^2)x+4\sqrt{3}Sx+3(-a^2+b^2+c^2)+4\sqrt{3}S)(8\sqrt{3}Sx+3(-a^2-b^2+c^2)-4\sqrt{3}S),$$
$$ab(3(a^2-b^2-c^2)x+4\sqrt{3}Sx+3(-a^2+b^2+c^2)+4\sqrt{3}S)(3(a^2-b^2+c^2)+4\sqrt{3}S)]]$$

Now we use a very useful property of trilinear coordinates : discriminant of 3x3 matrix made with the two trilinear coordinates of lines parallel and $[a, b, c]$ is zero.

Applying it to AF and DE we get the cubic equation with root x in $]0, 1[$:

$$24\sqrt{3}Sq_c^2x^3+$$
$$6((q_a^2-2q_aq_b+q_b^2-2q_aq_c-2q_bq_c+q_c^2)+4\sqrt{3}S(q_a-q_b-2q_c))q_cx^2+$$
$$4(q_c(-3q_a^2+6q_aq_b-3q_b^2+6q_aq_c+6q_bq_c-3q_c^2)+2\sqrt{3}S(q_a^2-2q_aq_b+q_b^2-2q_aq_c+4q_bq_c+4q_c^2))x$$
$$-3q_a^3+9q_a^2q_b-9q_aq_b^2+3q_b^3+9q_a^2q_c-6q_aq_bq_c-3q_b^2q_c-9q_aq_c^2-3q_bq_c^2+3q_c^3+$$
$$4\sqrt{3}S(-q_a^2+2q_aq_b-q_b^2+2q_aq_c-4q_bq_c-q_c^2)=0$$

Making equation independent to scaling triangle ABC : change to parameters $u=\frac{q_a}{q_c}$, $v=\frac{q_b}{q_c}$ and $w=\frac{S}{qc\sqrt{3}}=\frac{1}{48}\sqrt{-u^2-v^2+2uv+2u+2v-1}$.

Cubic equation is simplified to :

$$f_{uv}(x)=x^3+(u-v-4w-2)x^2+(2v-16w^2+8w+1)x+2uw-2vw+8w^2-v-2w=0$$

In the (u, v) cartesian plane, maximizing (resp. minimizing) curve of $g_0(u, v)=f_{uv}(0)$ (resp. $g_1(u, v)=f_{uv}(1)$) is a branch of conic in green (resp. blue) color under (resp. over) the dashed line.

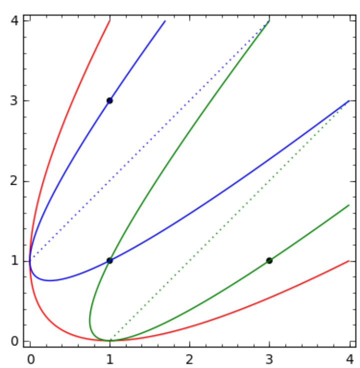

Figure 3: Signs of $f_{uv}(0)$ and $f_{uv}(1)$

Red conic is the border of valid values for (u, v) in order to define a real w.

On the branch of the conic, the value of the function is zero and negative everywhere else (resp. positive). Examples : $g_0(3,1)=g_0(3+\sqrt{2},2)=g_0(7/4,1/4)=0$, $g_1(1,3)=g_1(2,3+\sqrt{2})=g_1(1/4,7/4)=0$ and $g_0(1,1)=-1$, $g_1(1,1)=1$.

Except for (u,v) on the conics branches, opposite signs of $f_{uv}(0)$ and $f_{uv}(1)$ imply by continuity of f function that there is at least one real root x between 0 and 1.

The cubic equation has one or three real roots as given by the discriminant $\Delta(u,v)$ of the cubic. For (u,v) on the blue conic over the dashed line, discriminant is zero and there exists one triple root. Elsewhere discriminant is negative and one simple real root exists.

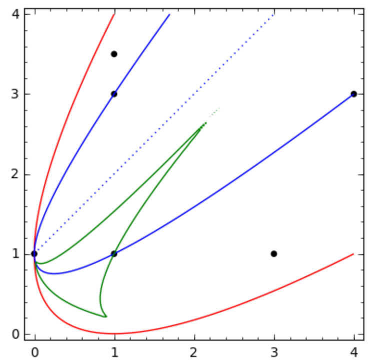

Figure 4: Cubic discriminant sign

Examples : $\Delta(1,3.5) = -168.369$, $\Delta(4,3) = -37152$, $\Delta(1,1) = -1674$, $\Delta(3,1) = -12960$ and $\Delta(1,3) = \Delta(0,1) = 0$.

Finding a real root of the cubic equation is not sufficient to get a valid dissection. For example with the 7:8:13 triangle, the computed F point for the real root $x = 0.301687148236$ is outside the ABC triangle.

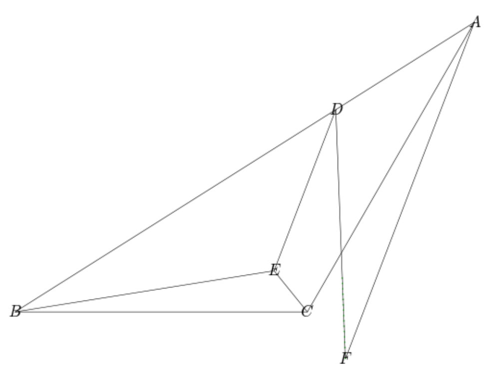

Figure 5: Triangle 7:8:13 "A too much sharp" has no dissection

Rotating ABC to a 13:8:7 or 13:7:8 or 7:13:8 triangle is no help because in that case $x = 0$ or $x = 1$ is the real root.

Triangle is "not too much A-sharp" when F is lying in the "-" area where the validity function $h_{uv}(x)$ is negative.

$$h_{uv}(x) = (x - x_1)(x - x_2)$$

with

$$x_1 = \frac{1}{2}(-\frac{q_a}{q_c} + \frac{q_b}{q_c} + 1) - \frac{2S}{q_c\sqrt{3}} = \frac{1}{2}(-u + v + 1) - 2w$$

and

$$x_2 = 1 + \frac{4S}{q_c\sqrt{3}} = 1 + 4w > 1$$

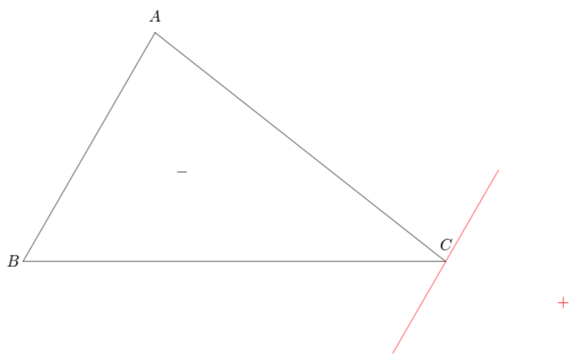

Figure 6: Sign of validity function

Validity function is deduced from trilinear coordinates of F and equation of line parallel with AB through C. Function is negative for x in $]x_1, x_2[$ and since $0 < x < 1$ we already have x in $]0, x_2[$. Hence, validity condition is $x > x_1 = \frac{2S}{q_c}(\cot A - \frac{1}{\sqrt{3}})$

When $\cot A < \frac{1}{\sqrt{3}}$ then $x_1 < 0 <= x$ and for sure triangle ABC has a valid dissection when x is different from 0 and 1. Family of "not too much A-sharp" include all triangles with that internal angle property and among them the A-rectangular and A-obtuse triangles.

The whole family of "not too much A-sharp" triangles having a five 120 degrees triangles dissection is defined by the equation $f_{uv}(x) = 0$ having real root different of 0 and 1, satisfying $x > x_1$.

3 Epilogue

You might wonder : what purpose for a new cyclic hinged dissection of a triangle ?

It is clearly related to hinged dissection of polygonal shapes in computational geometry and a kind of small follow up of [2] Erik D. Demaine's lecture about it. Hinged dissections can be used in designing things which can morph using hinges rotations.

In the lecture, it is explained how to fold or unfold a polygonal chain with slender adornments but... the problem of unfolding to convex a cyclic chain remains "open".

With dissection displayed in this paper, we have an unfolding to convex cyclic polygonal chain as showed in the Addendum chapter.

A simple dissection of a triangle ABC into three 120 degrees triangles AFB, BFC, CFA, using F first Fermat point, is not useful. The cyclic chain ABC made of segments opposite to F is already convex (cyclic).

Future work : dissect other polygonal shapes using the same idea of model dissection. For example a Blanche's dissection of a square is a model for dissection of squares.

4 Addendum : unfolding

When valid dissection exists, triangle ABC can be unfolded to a cyclic quadrilateral where two edges of the cyclic pentagonal chain are aligned.

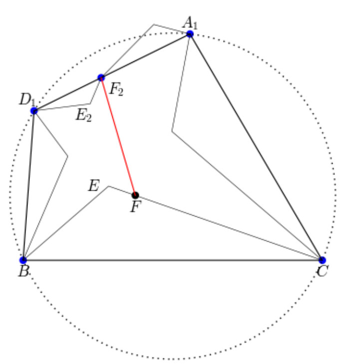

Figure 7: Unfolding to a cyclic quadrilateral

Folding or unfolding is simply done by stretching the FF_2 segment : rotations of four triangles triangles occur around hinges.

Apex of triangles are moving on circular arc.

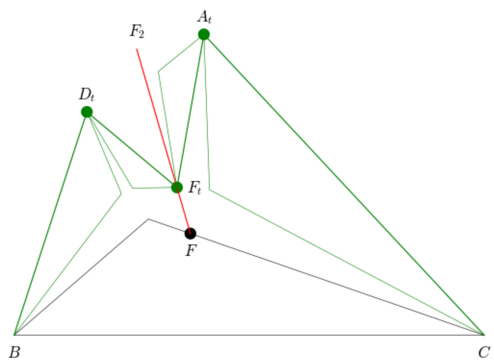

Figure 8: Intermediary configuration

The unfolding can be viewed as two simultaneous unfoldings : one for the open polygonal chain with slender adornments EDBF and the other one for EACF.

5 Acknowledgment

- MIT Open Course video (Erik D. Demaine, Lecture 14) about hinged dissections was source for the computational geometry content of this paper.
- Software [4] available at [3] cloud web site has been used to produce current paper (math computation and TeX editing).

References

[1] Zak, Andrzej. "Dissection of a triangle into similar triangles." Discrete & Computational Geometry 34.2 (2005): 295-312.

[2] Erik D. Demaine, *Lecture 14 : Hinged Dissections*, 2012, https://youtu.be/kRD_u8AUlwk

[3] Cocalc, Collaborative Calculation in the Cloud, aka "the SageMath Cloud" https://cocalc.com

[4] SageMath, the Sage Mathematics Software System (Version 8.0), The Sage Developers, 2017, http://www.sagemath.org