EXERCISES ON PERFECTOID SPACES

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Throughout, let p be a fixed prime.

1. Perfect rings and schemes

Let φ denote the absolute Frobenius on any ring or scheme over $\mathbb{F}_p$.

(1) Let $f : Y \to X$ be a morphism of perfect schemes.
   (a) Prove that f is formally unramified.
   (b) Prove the stronger assertion that the cotangent complex $L_{Y/X}$ is zero (as an object of the derived category of $\mathcal{O}_Y$-modules).

(2) (from [6, Lemma 3.2.12]) Let R be a perfect ring. Prove that any finitely presented R-module admitting an isomorphism with its φ-pullback is projective. Hint: consider the Fitting ideals of M.

(3) Prove that a morphism of perfect schemes which is locally of finite presentation is étale. Hint: use the vanishing of the cotangent complex to verify that the morphism is both smooth (see [9, Tag 01V4]) and unramified.

(4) Let $f : Y \to X$ be a flat surjective morphism of schemes. Show that if Y is perfect, then so is X. In particular, if f is also locally of finite presentation, then it is étale.

(5) Give an example of a morphism $R \to S$ of perfect rings which is injective and of finite type, but not of finite presentation. Hint: start with the normalization of a reducible scheme, then make everything perfect.

(6) Give an example of a reduced scheme over $\mathbb{F}_p$ whose residue fields are all perfect, but which is not itself perfect. Hint: take Spec of a suitable subring of a perfect valuation ring.

2. Nonarchimedean fields and Banach spaces

Convention: all nonarchimedean fields we consider have nontrivial norm.

(1) Let L/K be a finite extension of nonarchimedean fields. Prove that the sequence

$$0 \to \mathfrak{o}_K \to \mathfrak{o}_L \to \mathfrak{o}_L/\mathfrak{o}_K \to 0$$

of $\mathfrak{o}_K$-modules is almost split.

(2) Let K be a nonarchimedean field. Let V be a Banach space over K containing a dense subspace of at most countable dimension (such a Banach space is said to be *separable*). Prove that there exist elements $\mathbf{e}_1, \mathbf{e}_2, \dots$ such that the map

$$(a_1, a_2, \dots) \mapsto \sum_{n=1}^{\infty} a_n \mathbf{e}_n$$

defines an isomorphism of V with the space of null sequences over K. Such a sequence is called a *Schauder basis* of V over K. (This statement is famously false in the archimedean case, by work of Per Enflo.)

(3) Let K be a nonarchimedean field. Let $V_1 \to V_2$ be an inclusion of Banach spaces over K with V_1 being finite-dimensional. Prove that this inclusion splits. Hint: reduce to the case where V_2 is separable.

Convention: all nonarchimedean Banach rings we consider are commutative.

3. Comparison of Huber and Banach rings

(1) Let A be a Banach ring with norm $|\bullet|$.
   (a) Prove that for each $x \in A$, the limit $\lim_{n\to\infty} |x^n|^{1/n}$ exists. This limit, viewed as a function of x, is caled the *spectral seminorm* associated to (the given norm on) A.
   (b) Prove that the norm and the spectral seminorm coincide if and only if the norm is *power-multiplicative*: for all $x \in A$ and all positive integers n, $|x^n| = |x|^n$.
   (c) Let $|\bullet|'$ be a second norm on A which is *metrically equivalent* to A, that is, there exist $c_1, c_2 > 0$ such that

$$c_1 |x| \leq |x|' \leq c_2 |x| \qquad (x \in A).$$

   Prove that the spectral norms associated to the two norms coincide.
   (d) By contrast, let A be a nonarchimedean field. Observe that for each $c > 1$, the c-th power of the given norm is again a multiplicative norm defining the topology of A, but the resulting spectral seminorms are all distinct.

(2) In this exercise, we show that every Huber ring can be noncanonically promoted to a Banach ring. Let A be a (not necessarily Tate) Huber ring, let A_0 be a ring of definition of A, and let I be an ideal of definition of A_0.
   (a) Prove that for each $x \in A$, there exists some $n \in$ such that $xI^m \subseteq I^{m+n}$ for all $m \in$ for which $m, m+n \geq 0$ (interpreting I^0 as A_0). Hint: remember that A_0 is open in A and I is finitely generated as an ideal.
   (b) For $x \in A$, let $|x|$ be the infimum of e^{-n} over all integers n as in (a). Show that for all $x, y \in A$,
      (i) $|x| = 0$ if and only if $x = 0$;
      (ii) $|1| = 1$;
      (iii) $|xy| \leq |x||y|$;
      (iv) $|x + y| \leq \max\{|x|, |y|\}$;
      (v) for n a nonnegative integer, $|x| \leq e^{-n}$ if and only if $x \in I^n$.
   (c) Conclude that $|\bullet|$ is a submultiplicative norm defining the topology of A, and so gives A the structure of a Banach ring.

(3) In the other direction, let A be a Banach ring.
   (a) Show that if A is Tate, then the underlying topological ring of A is a Huber ring, and that the norm on A can be recovered by promotion for a suitable choice of a ring and ideal of definition.
   (b) Optional: extend (a) to the case where the topologically nilpotent elements of A generate the unit ideal.
   (c) Open: find a counterexample to (a) in the absence of any hypothesis on A. The difficulty in showing that A is Huber is to find a *finitely generated* ideal of definition.

(4) Let A be a Banach ring obtained by promotion from a Huber ring.

(a) Show that every bounded multiplicative seminorm on A is a continuous valuation on A which is bounded by 1 on A^+. We thus have a natural map $(A) \to \mathrm{Spa}(A, A^+)$.
(b) Prove that the images of
(b) Prove that if A is Tate, then the map is injective.

(5) Let A be a Banach ring whose underlying topological ring is a Huber ring. Consider the following conditions on A.
(i) The set A° of power-bounded elements is bounded; that is, A is *uniform* as a Huber ring.
(ii) The spectral seminorm is metrically equivalent to A; that is, A is *uniform* as a Banach ring.
(iii) The spectral seminorm defines the topology of A.
Now prove the following statements.
(a) If the norm on A is obtained by promotion from some ring and ideal of definition, then (i) and (ii) are equivalent.
(b) If A is Tate, then (i), (ii), and (iii) are equivalent. Hint: use the open mapping theorem.

4. More on Huber and Banach rings

(1) Let A be the quotient of the infinite Tate algebra $\mathbb{Q}_p\{T, U_1, V_1, U_2, V_2, \dots\}$ by the closure of the ideal $(TU_1 - pV_1, TU_2 - p^2V_2, \dots)$.
(a) Show that A is uniform.
(b) Show that T is not a zero-divisor in A.
(c) Show that the ideal TA is not closed in A.

(2) Prove that the Banach ring $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ is not uniform.

(3) Let (A, A^+) be a Huber pair.
(a) Show that the property of (A, A^+) being stably uniform depends only on A, not on A^+.
(b) Show that the property of (A, A^+) being sheafy depends only on A, not on A^+. Hint: use the fact that every covering is refined by some standard rational covering.

(4) Let $f : A \to B$ be an injective morphism of Banach rings.
(a) Show that if the image of f is closed, then f is strict. Hint: use the open mapping theorem.
(b) Show that if the image of f is closed and B is uniform, then so is A.
(a) Let K be an algebraically closed nonarchimedean field and put $A = K\{T^{1/p^\infty}\}$. Give a classification of rational subspaces of A analogous to the classification of rational subspaces of the closed unit disc (i.e., each one is a closed disc minus finitely many open subdiscs).
(b) Let

Convention: all nonarchimedean Banach rings we consider are commutative and contain a topologically nilpotent unit.

5. Rigid analytic geometry

(a) Let C a smooth, proper, geometrically irreducible curve of genus at least 2 over $\mathbb{Q}_p$. Let Y be a subspace of the analytification of E whose complement is isomorphic to an open disc. Prove that $\mathcal{O}^+$ is not acyclic on Y. Hint: for $\overline{C}$ the reduced curve, show that $H^1(Y, \mathcal{O}^+)$ surjects onto $H^1(\overline{C}, \mathcal{O})$.

6. Perfectoid fields

(a) Let K be a nonarchimedean field of residue characteristic p. Assume that the value group of K is p-divisible, the residue field of K is perfect, and K is *spherically complete* (every decreasing sequence of balls in K has nonempty intersection, even if the radii do not tend to 0). Prove that K is a perfectoid field.

(b) Let K be a discretely valued field of mixed characteristics with perfect residue field. Let L be an algebraic extension of K. Coates and Greenberg say that L/K is *deeply ramified* if for every finite extension L' of L, the map $\mathrm{Trace} : \mathfrak{m}_{L'} \to \mathfrak{m}_L$ is surjective. Prove that L/K is deeply ramified if and only if the completion of L is a perfectoid field.

(c) Prove that the completion of any algebraic extension of K which is *strictly arithmetically profinite* in the sense of Fontaine–Wintenberger is perfectoid. A stronger result is due to Fesenko [3]: any *arithmetically profinite* extension is deeply ramified in the sense of Coates–Greenberg, and hence has perfectoid completion by the previous exercise.

7. Perfectoid rings

We use Fontaine's definition of perfectoid rings (i.e., the "Fontaine perfectoid rings" of [7, §3.3]) unless otherwise qualified.

(a) Prove that a perfectoid ring is noetherian if and only if it is a finite direct product of perfectoid fields. Hint: first check the analogous statement about perfect rings.

(b) Let A be a perfectoid ring. Prove that A is *seminormal* in the sense of Swan: the map

$$A \to \{(y, z) \in A^2 : y^3 = z^2\}, \qquad x \mapsto (x^2, x^3)$$

is a bijection.

(c) Let K be a perfectoid field.
  (a) Define the *Newton polygon* of a nonzero element of $K\{T^{1/p^\infty}\}$ and show that it has the expected behavior under multiplication.
  (b) Prove that every irreducible element of $K\{T^{1/p^\infty}\}$ has only one slope in its Newton polygon (that is, elements of $K\{T^{1/p^\infty}\}$ admit factorizations by slopes). Hint: one approach is to apply the "master factorization theorem" from [4, Chapter 2].
  (c) Let $x, y \in K\{T^{1/p^\infty}\}$ be two elements, each of whose Newton polygons has a unique slope. Prove that if these two slopes differ, then the ideal (x, y) is trivial.

(d) Prove that $K\{T^{1/p^\infty}\}$ is a *Bézout domain*, i.e., an integral domain in which every finitely generated ideal is principal. Hint: exhibit a form of Euclidean division for two elements whose Newton polygons each have only one slope.

(e) Prove that the multiplication map

$$K\{T^{/1p^\infty}\} \times K\{T^{-1/p^\infty}\} \to K\{T^{\pm 1/p^\infty}\}$$

is surjective. Hint: see the hint for (b).

(f) Deduce that $K\{T^{\pm 1/p^\infty}\}$ is also a Bézout domain.

(d) Suppose $p > 2$, let K be an algebraically closed perfectoid field, and put $A = K\{T^{1/p^\infty}\}$ and $B = A[T^{1/2}]$. Let $x \in \operatorname{Spa}(A, A^\circ)$ be the pullback of the valuation on K along the projection

$$A \to A/(T, T^{1/p}, \dots) \cong K;$$

this point has a unique preimage $y \in \operatorname{Spa}(B, B^\circ)$.

(a) Observe that $\operatorname{Spa}(B, B^\circ) - \{y\} \to \operatorname{Spa}(A, A^\circ) - \{x\}$ is finite étale, and deduce that $\operatorname{Spa}(B, B^\circ) - \{y\}$ is a perfectoid space.

(b) Prove that B is not a perfectoid ring. In particular, B is an example of a finite algebra over a perfectoid ring which is not perfectoid.

(c) Put $B' = K\{T^{1/(2p^\infty)}\}$. Prove that the map $\operatorname{Spa}(B', B'^\circ) \to \operatorname{Spa}(B, B^\circ)$ is a homeomorphism and matches up rational subspaces.

(d) Pick $\lambda, \mu \in \mathfrak{o}_K$ with $\mu \neq 0$. Let $(B, B^\circ) \to (C, C^+)$ be the rational localization corresponding to the subspace

$$\{v \in \operatorname{Spa}(B', B'^\circ) : v(T^{1/(2p)} - \lambda) \geq v(\mu)\}.$$

Prove that C is uniform.

(e) By imitating the previous step, prove that B is stably uniform. In particular, B is an example of a stably uniform Banach ring which is not perfectoid but whose residue fields are all perfectoid.

(e) (from [7, Theorem 3.3.24]) Let G be a finite group acting on a perfectoid ring A.

(a) Suppose that G is a p-group. Prove that the fixed subring A^G is perfectoid. Hint: given $x \in A^G$, approximate it with a Teichmüller lift in A, then take the *geometric* mean over G-conjugates.

(b) Extend (a) to arbitrary G. Hint: apply (a) to a p-Sylow subgroup of G, then take the *arithmetic* mean over cosets.

(f) Let K be a perfectoid field. Let $I \subset (0, +\infty)$ be a closed interval. Let A be the completion of $K(T^{1/p^\infty})$ for the supremum of the ρ-Gauss norms for all $\rho \in I$.

(a) Prove that A is a perfectoid ring.

(b) Prove that A is not a field. This observation is a key step in the proof that every perfectoid ring which is a field is a perfectoid field [5].

(g) Let A be a perfectoid ring with tilt $A^\flat$.

(a) Prove that any finite reduced $A^\flat$-algebra is perfect and untilts to a finite perfectoid A-algebra.

(b) Prove that any finite *perfectoid* A-algebra B tilts to a finite $A^\flat$-algebra. Hint: first show that B is generated as an A-module by Teichmüller lifts. Then show that B° is almost finitely generated over A°.

(h) (inspired by [8]) Let K, L be finite extensions of $\mathbb{Q}_p$ with absolute Galois groups G_K, G_L. Let $\mathbb{C}_K, \mathbb{C}_L$ be completed algebraic closures of K, L. Suppose that there exists an isomorphism $\mathbb{C}_K^\flat \cong \mathbb{C}_L^\flat$ of perfectoid fields which is equivariant with respect to an isomorphism $G_K \cong G_L$ of profinite groups. Prove that these data arise from some isomorphisms $K \cong L$, $\mathbb{C}_K \cong \mathbb{C}_L$. Hint: use local class field theory to construct the isomorphism $K \cong L$ of additive topological groups, and likewise for corresponding finite extensions. Then match up the multiplicative structures by comparing $\mathfrak{o}_{\mathbb{C}_K}/(p)$ with $\mathfrak{o}_{\mathbb{C}_L}/(p)$.

(i) Let A be a uniform Banach ring over a perfectoid field satisfying "almost purity" in the following sense: for every finite étale A-algebra B, the ring map $A^\circ \to B^\circ$ is almost finite étale. Prove that A is perfectoid. (The result remains true without working over a perfectoid field, but the proof is slightly more technical.)

8. Perfectoid spaces

(a) (from [10]) Prove that any adic space which is smooth over a perfectoid space is stably uniform, and hence sheafy.

(b) Prove that any universal homeomorphism of perfectoid spaces is an isomorphism. Hint: use the seminormality of perfectoid rings.

(c) (from [2]) Let K be a perfectoid field. Define the *projectivoid line* over K by glueing the spaces $\mathrm{Spa}(K\{T^{1/p^\infty}\}, K^\circ\{T^{1/p^\infty}\})$ and $\mathrm{Spa}(K\{T^{-1/p^\infty}\}, K^\circ\{T^{-1/p^\infty}\})$ along $\mathrm{Spa}(K\{T^{\pm 1/p^\infty}\}, K^\circ\{T^{\pm 1/p^\infty}\})$. Prove that the Picard group of this space is isomorphic to $\mathbb{Z}[1/p]$. It is not known whether this generalizes to the projectivoid n-space for arbitrary n, or whether every vector bundle on the projectivoid line is a direct sum of line bundles.

(d) Let E be an elliptic curve over a perfectoid field. Show that the inverse limit of E under the multiplication-by-p morphism is similar to a perfectoid space. This is also true for abelian varieties.

9. Fargues-Fontaine curves

Throughout what follows, let F be a fixed algebraically closed perfectoid field of characteristic p, and let X be the associated Fargues-Fontaine curve. For any $n \in \mathbf{Z}$, let $\mathcal{O}(n)$ denote the standard line bundle on X of degree n.

For the definition of a *Farey sequence*, see Wikipedia.

(a) Let $\mathcal{E}$ be a vector bundle on X.

  (a) Prove that for some large n, we can find an injection $i : \mathcal{O}^{\mathrm{rank}(\mathcal{E})} \hookrightarrow \mathcal{E}(n) \stackrel{def}{=} \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(n)$, with (necessarily) torsion cokernel. Show moreover that for any given closed point $x \in X$, we can choose i such that $\mathrm{Supp}\,\mathrm{coker}(i) = x$. (Hint: Use the (difficult) fact that $X \setminus x$ is the spectrum of a PID.)

  (b) Prove that the spaces $H^0(X, \mathcal{E}), H^1(X, \mathcal{E})$ are Banach-Colmez spaces. (Hint: Use (a) together with the fact that Banach-Colmez spaces form an abelian category.)

  (c) Verify the Riemann-Roch formula:

$$\dim H^0(X, \mathcal{E}) - \dim H^1(X, \mathcal{E}) = (\deg(\mathcal{E}), \mathrm{rank}(\mathcal{E})).$$

(Hint: Reduce to line bundles by induction on $\mathrm{rank}(\mathcal{E})$, using the fact that dimension of Banach-Colmez spaces is componentwise-additive in short exact sequences. Then show that the formula holds for a line bundle $\mathcal{L}$ iff it holds for $\mathcal{L}(1)$; this reduces one to considering $\mathcal{L}$ of degree zero, in which case $\mathcal{L} \simeq \mathcal{O}$.)

(d) i. Prove that both the principal and residual dimension of $H^0(X, \mathcal{E})$ are nonnegative.
ii. Prove that if $H^1(X, \mathcal{E}) \neq 0$, then its principal dimension is positive and its residual dimension is negative.
(Hint for both: Use the classification.)

(e) (Bonus.) Deduce (d) from various results in Colmez's original paper, without appealing to the classification. Then turn the tables and reprove the classification using (b), (c), and (d).

(b) Let $\mathcal{E}$ be a semistable vector bundle on X of rank ≥ 2 and slope ≥ 0. Show that $\mathcal{E}$ has a nowhere-vanishing section.

(c) Given a vector bundle $\mathcal{E}$, we say a subbundle $\mathcal{E}^+$ is *saturated* if $\mathcal{E}/\mathcal{E}^+$ is a vector bundle. (Since X is Noetherian, regular and one-dimensional, this holds iff $\mathcal{E}/\mathcal{E}^+$ is torsion-free.) Show that for any $n \geq 0$, there is a saturated line subbundle $\mathcal{L} \subset \mathcal{O}^2$ such that $\mathcal{L} \simeq \mathcal{O}(-n)$. (Hint: By twisting, this reduces to finding a saturated subbundle $\mathcal{O} \simeq \mathcal{L}' \subset \mathcal{O}(n)^2$, at which point you can use the previous exercise.)

(d) Let $0 \leq \lambda < \mu \leq 1$ be reduced rationals; we say they are *Farey neighbors* if they occur as consecutive entries in some Farey sequence F_N. Let N be the minimal such integer for which this occurs, and let ν^- (resp. ν^+) be the entry in F_N immediately before λ (resp. immediately after μ). Prove that any nonzero map $\mathcal{O}(\lambda) \to \mathcal{O}(\mu)$ is either
  (i) injective with cokernel $\simeq \mathcal{O}(\nu^+)$, or
  (ii) surjective with kernel $\simeq \mathcal{O}(\nu^-)$,
according to whether $\mathrm{denom}(\lambda) < \mathrm{denom}(\mu)$ or $\mathrm{denom}(\mu) < \mathrm{denom}(\lambda)$.

(e) Prove that if $0 \leq \lambda < \mu \leq 1$ are Farey neighbors, then any bundle $\mathcal{F}$ sitting in a nonsplit extension

$$0 \to \mathcal{O}(\lambda) \to \mathcal{F} \to \mathcal{O}(\mu) \to 0$$

is isomorphic to $\mathcal{O}(\eta)$, where η is the mediant of λ and μ.

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