The Circumference of a Circle

To find the circumference C of a circle, we will double the arc length $L = \frac{C}{2}$ of the associated semicircle. Recall that a circle with radius r in the plane is given by the equation

$$x^2 + y^2 = r^2.$$

Solving for y, we find that the semicircle with radius r is given by

$$y = \sqrt{r^2 - x^2},$$

which is defined on $[-r, r]$ and has derivative

$$y' = \frac{x}{\sqrt{r^2 - x^2}}.$$

Using the arc length formula, we find that

$$\begin{aligned}
L &= \int_{-r}^{r} \sqrt{1 + (y')^2}\ dx \\
&= \int_{-r}^{r} \sqrt{1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2}\ dx \\
&= \int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\ dx \\
&= \int_{-r}^{r} \sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}\ dx \\
&= \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\ dx.
\end{aligned}$$

Continue with a trigonometric substitution: let $x = r\sin\theta$, so $dx = r\cos\theta\ d\theta$ and $\sqrt{r^2 - x^2} = r\cos\theta$. Then

$$\begin{aligned}
L &= \int_{x=-r}^{x=r} \frac{r}{\sqrt{r^2 - x^2}}\ dx \\
&= \int_{\theta=\sin^{-1}(-1)}^{\theta=\sin^{-1}(1)} \frac{r}{r\cos\theta} r\cos\theta\ d\theta \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} r\ d\theta \\
&= \big[r\theta\big]_{-\pi/2}^{\pi/2} \\
&= r\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= r\pi.
\end{aligned}$$

This means that the circumference of the circle wuith radius r is

$$C = 2L = 2\pi r.$$