Modular motifs

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1 Introduction

Canonical disclaimer: I am not an expert!

Goal: I will describe Tony Scholl's [Invent. **100**, 419–430 (1990)] theorem:

$$S_k(N) \mapsto (\text{Chow motif over } \mathbf{Q})$$

Scholl also associates a Grothendieck motif to each newform f.

> "To construct the motif means to isolate a constituent of the cohomology of a variety using algebraic cycles."

2 Connectedness of Spec$\mathbf{T}$

First, I'll relay a question. Let

$$\mathbf{T} \subset \mathrm{End}(S_k(\Gamma_0(p)))$$

be the Hecke algebra.

Question (Ash, Mazur): Is Spec$\mathbf{T}$ connected?
When $k = 2$, *yes*, by Prop.II.10.2 of Mazur's *Modular curves and the Eisenstein ideal* paper.
Open problem when $k > 2$, but numerical evidence is encouraging: a related graph is connected for $k \leq 8$ and $p \leq 53$.
Mazur's proof involves Ribet's correspondence between irreducible components of Spec($\mathbf{T}$) and isogeny factors of $J_0(p)$; reducibility of the Hecke algebra quickly translates into a decomposition of the canonical principal polarization, a contradiction. Is there a higher weight motific analogue of Mazur's proof?

> "Well, the reason why the question is particularly intriguing is that we DON'T seem to have any hope of a proof (of connectedness for weight $k > 2$ Hecke algebras) that mimics the proof for weight 2. The higher weight motives just don't (YET...) have the precise algebro-geometric structure that come from principally polarized abelian varieties. So one would need an utterly different kind of proof, (if the connectedness property is true... which, after your calculations looks possible)."
> Mazur, email, yesterday.

What sort of structure *do* we have??

3 Products of universal elliptic curves

Fix integers $N, k \geq 3$, and set $w = k - 2$ (this is the opposite of the notation in Scholl, but I feel more comfortable with it). The case $k = 2$ is $J_0(N)$, which was already done by Eichler and Shimura.

$$\begin{tikzcd}
M_N/\mathbf{Q} \arrow[r, equal] \arrow[d, hook] & \{\text{elliptic curves } E, \text{ with } (\mathbf{Z}/N)^2 \hookrightarrow E\} \arrow[d, hook, "\text{smooth compactification}"] \\
\overline{M}_N = M_N \cup M_N^\infty & (\text{generalized elliptic curves + level structure})
\end{tikzcd}$$

- X_N is the universal elliptic curve over M_N
  $\overline{X}_N$ is the generalized elliptic curve; it is a smooth and proper $\mathbf{Q}$-scheme
  $\overline{X}_N^*$ is the Néron model of $X_N/\overline{M}_N$ (smooth locus of $\overline{\pi}$.)

$$\begin{tikzcd}
X_N \arrow[d, "\pi"'] & \overline{X}_N \arrow[d, "\overline{\pi}"'] & \\
& & \overline{X}_N^* \arrow[ul, hook'] \arrow[dl] \\
M_N \arrow[r, hook, "j"] & \overline{M}_N &
\end{tikzcd}$$

- The product $\overline{X}_N \times_{\overline{M}_N} \cdots \times_{\overline{M}_N} \overline{X}_N$ is singular over $\mathbf{Q}$ when there is more than one factor.

$$\begin{tikzcd}
& \overline{\overline{X}}_N^w \arrow[d] & \text{Deligne's canonical desingularization} \\
X_N^w \arrow[d, equal] & \overline{X}_N^w \arrow[r, equal] \arrow[d, "\overline{\pi}_w"'] & \overline{X}_N \times_{\overline{M}_N} \cdots \times_{\overline{M}_N} \overline{X}_N \ (w \text{ factors}) \\
\overline{\pi}_w^{-1}(M_N) & \overline{M}_N &
\end{tikzcd}$$

$$\begin{aligned}
\overline{X}_N^{w,\infty} &:= \overline{X}_N^w - X_N^w \\
\overline{\overline{X}}_N^{w,\infty} &:= \overline{\overline{X}}_N^w - X_N^w
\end{aligned}$$

4 Automorphisms of Deligne's desingularization

The level-N structure on the universal elliptic curve X_N over M_N:

$$(\mathbf{Z}/N)^2 \hookrightarrow X_N \quad (\text{over } M_N).$$

By the Néron mapping property, this extends to a morphism of Néron models:

$$(\mathbf{Z}/N)^2 \times \overline{M}_N \hookrightarrow \overline{X}_N^* \hookrightarrow \overline{X}_N,$$

so $(\mathbf{Z}/N)^2$ acts by translation on $\overline{X}_N$.

Also $\mu_2 = \{\pm 1\}$ acts on $\overline{X}_N$, so

$$(\mathbf{Z}/N)^2 \rtimes \mu_2 \quad \text{acts on } \overline{X}_N.$$

(Both groups sit in $\mathrm{Aut}(\overline{X}_N)$ and $(\mathbf{Z}/N)^2$ has index 2.)

- Let Σ_w be the symmetric group on $\{1, 2, \ldots, w\}$.
- The **wreath product** of $(\mathbf{Z}/N)^2 \rtimes \mu_2$ by Σ_w:

$$\begin{aligned}\Gamma_w &= ((\mathbf{Z}/N)^2 \rtimes \mu_2) \wr \Sigma_w \\ &= ((\mathbf{Z}/N)^2 \rtimes \mu_2)^w \rtimes \Sigma_w.\end{aligned}$$

The wreath product is the following set of permutations θ of $\overline{X}_N \times \{1, 2, \ldots, w\}$:

$$\{\theta : \theta(a, b) = (\gamma_b(a), \eta(b)) : \gamma_b \in (\mathbf{Z}/N)^2 \rtimes \mu_2, \eta \in \Sigma_w\},$$

where $a \in \overline{X}_N$, $b \in \{1, \ldots, w\}$, and γ_b is chosen independently for each b. Automorphisms of $\overline{X}_N \times \{1, 2, \ldots, w\}$ induce automorphisms of $\overline{X}_N^w$; and of $\overline{\overline{X}}_N^w$ (because the desingularization is canonical).

5 The projector Π_ε

$$\begin{tikzcd} \Gamma_w \arrow[rr, "\varepsilon"] \arrow[d, equal] & & \{\pm 1\} \\ ((\mathbf{Z}/N)^2 \rtimes \mu_2)^w \rtimes \Sigma_w & & \end{tikzcd}$$

$$((a_i),(b_i),\sigma) \longmapsto \left(\prod_{i=1}^w b_i\right) \cdot \mathrm{sign}(\sigma)$$

Associate to ε a projector in the group ring:

$$\Pi_\varepsilon \in \left(\mathbf{Z}\left[\frac{1}{2N\cdot(w!)}\right]\right)[\Gamma_w].$$

We have

$$\Pi_\varepsilon = \frac{1}{\#\Gamma_w}\sum_{g\in\Gamma_w}\varepsilon(g)g$$

because

$$\left(\sum \varepsilon(g)g\right)^2 = \#\Gamma_w \sum \varepsilon(g)g,$$

which can be seen by considering how $g \in \ker(\varepsilon)$ and $g \notin \ker(\varepsilon)$ act on $\sum_{h\in\ker(\varepsilon)} h$ and $\sum_{h\notin\ker(\varepsilon)} h$.

Define

$$V(\varepsilon) := \Pi_\varepsilon(V) \subset V.$$

6 Parabolic cohomology groups

Cohomological realization of cusp forms (Eichler-Shimura correspondence type stuff):

$$\begin{aligned} {}^{w}_{N}W_\ell &:= H^1_{\text{ét}}(\overline{M}_N \otimes \overline{\mathbf{Q}}, j_*\mathrm{Sym}^w((R^1\pi_*)\mathbf{Q}_\ell)) \\ &= \ker(H^1_{\text{ét}}(M_N \otimes \overline{\mathbf{Q}}, \mathrm{Sym}^w R^1\pi_*\mathbf{Q}_\ell) \\ &\qquad \to H^0_{\text{ét}}(M_N^\infty \otimes \overline{\mathbf{Q}}, \mathbf{Q}_\ell(-k-1))) \end{aligned}$$

Theorem 6.1 (Scholl). *For every prime ℓ,*

$$ {}^{w}_{N}W_\ell = H^*_{ét}(\overline{\overline{X}}{}^w_N)(\varepsilon)$$

and a similar statement for the Betti cohomology.

Though desingularizing totally "messes up" the variety, miraculously a careful study reveals that the ε part of cohomology remains under control. Most of Scholl's proof is devoted to understanding the desingularization of the compactification.

The pair

$$(\overline{\overline{X}}{}^w_N, \Pi_\varepsilon)$$

has a right to be called *the motif attached to the space of modular forms of weight k and level N*. Let ${}^{w}_{N}\mathcal{W}$ be the motif defined by $(\overline{\overline{X}}{}^w_N, \Pi_\varepsilon)$ in the category of *Chow* motifs over $\mathbf{Q}$ (rational equivalence).

7 Hecke operators on the motif ${}^{w}_{N}\mathcal{W}$

It is possible to decompose ${}^{w}_{N}\mathcal{W}$ in the category of Grothendieck motives [homological equivalence, tensor with $\mathbf{Q}(\dots a_n \dots)$]. "It should be possible to decompose ${}^{w}_{N}\mathcal{W}$ already in the category of Chow motives; but this seems very hard without assuming the standard conjectures." (???)

- Fix $p \nmid N$ a prime:

  $$M_{N,p} := \{(E, (\mathbf{Z}/N)^2 \text{ structure }, C \subset E[p] \text{ order } p)\}$$

  This is a curve over $\mathbf{Q}$.

- $$\begin{tikzcd} C \arrow[r, hook] & X_{N,p} \arrow[r] \arrow[d] & X_N \arrow[d] \\ & M_{N,p} \arrow[r] & M_N \end{tikzcd}$$

- $Q := X_{N,p}/C$ + level N structure induced from $X_{N,p}$.

  $$Q^w := Q \times_{M_{N,P}} \times \cdots \times_{M_{N,p}} Q \quad (w\text{-fold product})$$

- Consider

  $$\begin{tikzcd} X_N^w \arrow[d] & X_{N,p}^w \arrow[l, "\phi_1"'] \arrow[r, "\psi"] \arrow[d] & Q^w \arrow[r, "\phi_2"] \arrow[d] & X_N^w \arrow[d] \\ M_N & M_{N,p} \arrow[l] \arrow[r, equal] & M_{N,p} \arrow[r] & M_N \end{tikzcd}$$

  The first and third squares are cartesian.

- Define the **Hecke correspondence** T_p on X_N^w by

$$T_p = (\phi_1)_* \psi^* \phi_2^*.$$

  The correspondence T_p on $\overline{\overline{X}}{}_N^w$ is the closure of the graph of T_p in $\overline{\overline{X}}{}_N^w \times \overline{\overline{X}}{}_N^w$.

Proposition 7.1 (Scholl).

- *Γ_w on $H_B^*(\overline{\overline{X}}{}_N^w)$ commutes with T_p.*
- *Action of* transpose *of T_p on $H^0(\overline{\overline{X}}{}_N^w, \Omega^{k-1})$ is the classical Hecke operator T_p.*

So we have an endomorphism of motives:

$$T_p : {}_N^w\mathcal{W} \to {}_N^w\mathcal{W}$$