Introduction to modular motives

William Stein, UCB

November 23, 1999

1 Introduction

WARNING – These are informal notes, and I am not an expert!

Long-term goal: *Learn how to* compute *enough invariants of specific motives associated to modular forms of weight greater than* 2 *to verify the Bloch and Kato conjecture in a few special cases.*

As far as I know, the above has not been done for even a single example.

Questions:

1. What are motives? (Grothendieck, etc.)
2. What are modular motives? (Scholl)
3. How are L-functions related to motives? (Galois representations)
4. What did Bloch and Kato conjecture? (Grothendieck Feitschrift)
5. How can we compute examples? (????????)

2 Birch and Swinnerton-Dyer conjecture

Conjecture: Let A/K be an abelian variety over a number field K. Then

$$L(1) = \frac{\#\text{Ш}(A) \cdot \prod c_v \cdot \Omega_A \cdot \mathrm{Reg}}{A(K)_{\mathrm{tor}} \cdot A^\vee(K)_{\mathrm{tor}}}$$

where

- $L(1)$ is the *leading coefficient* of the L-series at 1 (that this is defined is part of the conjecture).
- $\text{Ш} = \ker(H^1(K, A) \to \prod_v H^1(K_v, A))$ measures failure of the local-to-global principle.
- c_v = local volumes at finite places = orders of component groups.
- Ω_A = product of volumes at infinite places with respect to basis of integral differentials. (I don't know how to define this when $\mathcal{O}$ has lots of units.)
- Reg = regulator = discriminant of canonical height pairing $A(K) \times A^\vee(K) \to \mathbf{R}$.
- $A(K)_{\mathrm{tor}}$ = torsion subgroup.

3 Modular abelian varieties

Eichler-Shimura: Let

$$f = q + a_2q^2 + a_3q^3 + a_4q^4 + a_5q^5 + \cdots$$

be a weight-2 modular eigenform with character ε. Let $\mathfrak{p}_f$ be the annihilator of f in the Hecke algebra. Shimura associated to f an abelian variety $A_f = J_1(N)/\mathfrak{p}_f J_1(N)$. Eichler-Shimura proved the following nontrivial theorem: *the characteristic polynomial of* Frob_p *on the ℓ-adic Tate module of A_f tensored with $\mathbf{Q}_\ell(\ldots a_n \ldots)$ is $X^2 - a_pX + p\varepsilon(p)$.*

See Brian Conrad's notes.

Writing down abelian varieties: In dimension greater than two, there appears to be no **satisfactory** way to "write down" an abelian variety. In contrast, the *modular* abelian varieties (up to isogeny) *can* be "written down" by giving the corresponding modular forms! For example,

$$f = q + aq^2 + (-a^2 - a + 3)q^3 + \cdots,$$

where $a^3 + a^2 - 3a - 1 = 0$, defines a three-dimensional quotient of the four-dimensional abelian variety $J_0(53)$. The other part of $J_0(53)$ is an elliptic curve, which we can write down, in the classical sense.

For many purposes the modular form is just as good a starting point as explicit equations.

4 The Bloch-Kato conjecture

Question: Do abelian varieties fit into a "universal" category of objects possessing L-functions, arithmetic invariants and BSD-like relations?

Idea of Bloch and Kato: Try to approach the invariants appearing in the BSD conjecture from the point of view of cohomology.

- The L-function packages together the characteristic polynomials of Frobenius acting on the Tate module (=ℓ-adic cohomology).
- Ш is a kernel of a map between Galois cohomology groups. (Mazur defined Ш using flat cohomology.)
- c_p and Ω are volumes with respect to a measure. Somehow these can be recovered via de Rham and Betti cohomology and a choice of lattice, but I don't understand this yet.
- Torsion (?)
- Regulator (?)

My understanding is that Bloch and Kato succeed, modulo "standard conjectures in K-theory."

5 Grothendieck Motives

Grothendieck dreamed up a category of motives while searching for a category $\underline{\mathbf{M}}$ equipped with a "universal cohomology functor"

$$H : \{\text{proj. nonsing. alg. var. over } k\} \to \underline{\mathbf{M}}.$$

This H should be univeral in the sense that any other (reasonable) cohomology theory, for example, étale cohomology, should factor through it.

Homomorphisms: Consider the category whose objects are algebraic varieties, but with the following weird morphisms:

$$\mathrm{Hom}(X, Y) := \frac{\{\text{algebraic cycles on } X \times Y \text{ of dim. } \dim X\}}{(\text{numerical equivalence})}$$

(Numerically equivalent cycles are indistinguishable from the point of view of the intersection pairing into .) These morphisms form a *group* so our first thought is: "abelian category!!!!"

Unfortunately, this category *is far from abelian.* There are lots of projectors, i.e., maps that equal their square, whose kernels are not algebraic varieties.

Motives: Formally "add in all kernels of projectors", to obtain a pseudo-abelian category. Motives are represented by pairs (X, π) where X is a k-variety and $\pi \in \mathrm{Hom}(X, X)$, $\pi^2 = \pi$.

6 Realizations of motives

Quote: Manin gives a remark of Herbert Read:

> Cézanne's method of painting was first to choose his visual apprehension of this motif; and in this process loose nothing of the vital intensity that the motif possessed in actual existence.

Remark: There is a special Cézanne exhibit at `http://metalab.unc.edu/wm/paint/auth/cezanne/`.

Realization: Cohomology functors on the category of algebraic varieties, such as étale, Hodge, etc., (should?) factor through the category of motives. Thus, given a motive M, there should be finite dimensional vector spaces $H_{\text{ét}}(M)$, $H_{\text{Hodge}}(M)$, etc. attached to M. These are the realizations of the motive M.

7 Modular motives

Let f be a modular eigenform of weight greater than 2.

Deligne: Associated a compatible system of ℓ-adic Galois representations to f satisfying a characteristic polynomial condition as in Eichler-Shimura.

Shimura: Using period integrals, Shimura constructs an abelian variety over $\mathbf{C}$ attached to f and equipped with action of higher-weight Hecke operators. It has no apparent arithmetic structure and have been ignored (?), as far as I can tell.

Scholl: Using Kuga-Sato varieties, Scholl constructs a motive that is naturally linked up with f. The Kuga-Sato varieties are desingularized compactifications of products of universal elliptic curves. I will describe Scholl's result more precisely next week.