Applications of Chebotarev Density Theorem to Cusp Forms

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Abstract

The goal of this article is to apply Chebotarev density theorem to obtain density result about irreducibility of the characteristic polynomial of the Hecke operators attached to the weight 24 cusp forms of level 1, $S_{24}(\Gamma_0(1))$.

1 Preliminary theorems

In this section, I will state the theorems I will use to prove the result mentioned in the abstract.

Theorem 1 (Chebotarev Density Theorem).

Theorem 2 (Ribet's Big Image Theorem).

Theorem 3 (Deligne).

2 Main Result and its Proof

Theorem 4. *Suppose $f \in S_{24}(\Gamma_0(1))$ is an eigenform. (i.e. $f = q + a_2q^2 + \dots$) and a_p is a root of* $\mathrm{charpoly}(T_p)$*. Then the set $\{p$ prime : $\mathrm{charpoly}(T_p)$ is reducible over $\mathbf{Q}\}$ has density 0.*

Proof. Let $f \in S_{24}(\Gamma_0(1))$ be an eigenform, such that $f = q + a^2q^2 + \dots$, and we know that a_2 is a root of $\mathrm{charpoly}(T_2)$ and it is irreducible. Let $K_f = \mathbf{Q}(\sqrt{D})$ be the quadratic field that is generated by the coefficients a_n of f.

Let
$$P_\ell = \{p \text{ prime} : \mathrm{charpoly}(T_p) \text{ is reducible over } \mathbb{F}_\ell\}.$$

Thus $P_\ell = \{p \text{ prime} : \overline{a_p} \in \mathbb{F}_\ell\}$ since a_p is a root of charpoly(T_p), which in turn is equal to $\{p \text{ prime} : \operatorname{tr}(\overline{\rho}(\mathrm{Frob}_p))) \in \mathbb{F}_\ell\}$ by Deligne's theorem, where $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K_f}/(\ell))$ is the mod ℓ Galois representation.

Note that $\overline{a_2} \in \mathbb{F}_\ell$ iff charpoly(T_2) splits in $\mathbb{F}_\ell[t]$ iff ℓ splits completely in $\mathcal{O}_{K_f}$. By Chebotarev density theorem, the set of primes which split completely in $\mathcal{O}_{K_f}$ has density 1/2. This says the density of inert primes is 1/2.

Since f is a non-CM form, Ribet's big image theorem says the projective mod ℓ Galois representation: $\mathrm{proj}(\overline{\rho}) : G_{\mathbf{Q}} \to \mathrm{PGL}_2(\mathcal{O}_{K_f}/(\ell))$ is surjective for all but finitely many primes ℓ. In particular, since there are infinitely many primes ℓ that are inert, we have

$$\mathrm{proj}(\overline{\rho}) : G_{\mathbf{Q}} \to \mathrm{PGL}_2(\mathbb{F}_{\ell^2})$$

is surjective for infinitely many inert primes ℓ.

Now let $G = \mathrm{Gal}(\overline{\mathbf{Q}}^{\ker(\mathrm{proj}(\overline{\rho}))}/\mathbf{Q})$, then $|G| = |\operatorname{Im}(\mathrm{proj}(\overline{\rho}))| = |\mathrm{PGL}_2(\mathbb{F}_{\ell^2})|$, by Galois theory.

Let

$$\mathcal{F}_c = \{p : [\mathrm{proj}(\overline{\rho}(\mathrm{Frob}_p))] = c\},$$

for each conjugacy class $c \in \mathrm{PGL}_2(\mathbb{F}_{\ell^2})$ whose trace is in $\mathbb{F}_\ell$. Here, $[\mathrm{proj}(\overline{\rho}(\mathrm{Frob}_p))]$ denotes the conjugacy class of $\mathrm{proj}(\overline{\rho}(\mathrm{Frob}_p))$.

Then by Chebotarev density theorem, we have the density of $\mathcal{F}_c$ equals $|c|/|G|$.

Now, $P_\ell = \{p : \operatorname{tr}(\overline{\rho}(\mathrm{Frob}_p)) \in \mathbb{F}_\ell\} \subset \cup_c \mathcal{F}_c$, where c runs through all conjugacy classes of $\mathrm{PGL}_2(\mathbb{F}_{\ell^2})$ with trace in $\mathbb{F}_\ell$. The union is disjoint and we have:

$$\text{density of } P_\ell \leq \sum_c \frac{|c|}{|G|}.$$

The following lemma calculates the number of conjugacy classes c and the size of each c:

Lemma 1. *There are three types of conjugacy classes c in $\mathrm{PGL}_2(\mathbb{F}_{\ell^2})$, which are:*

$$I, \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},$$

where $a \neq 1$.

Furthermore, with the restriction of trace, and determinant being in the ground field $\mathbb{F}_\ell$ there are $1, \ell-2, \ell-1$ conjugagcy classes of the 1st, 2nd, and 3rd type respectively and the size of c of type 1, 2, 3 are $1, \ell^2-1$, and $(\ell^2-1)(\ell^2)$ respectively.

Proof. With the restriction that trace and determinant are in the ground field, we have that $\mathbb{F}_{\ell^2}$ contains all eigenvalues and we may apply Jordan Canonical form, then we have the classification into the three types at once. It is obvious that there are only 1 class that contains I. For the 2nd type, since determinant lies in the ground field implies $a \in \mathbb{F}_\ell$, we have $\ell - 2$ choice because we are omitting 0 and 1. Finally, for the 3rd type, $2a \in \mathbb{F}_\ell$ implies $a \in \mathbb{F}_\ell$ and that left $\ell - 1$ choices for a by omitting 0. The size of c of 1st type is obviously 1 since everything commutes with I. By direct calculation, the set of matrices that commutes with a type 2 matrix and type 3 matrix looks like:

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \quad \begin{pmatrix} x & y \\ 0 & z \end{pmatrix},$$

respectively and there are $\ell^2 - 1$ and $(\ell^2 - 1)(\ell^2)$ of each respectively. This finishes the proof of the lemma. $\square$

By the lemma we have:

$$\sum_c \frac{|c|}{|G|} = \frac{1 + (\ell^2 - 1)(\ell - 2) + (\ell - 1)(\ell^2 - 1)(\ell^2)}{(\ell^2 + 1)(\ell^4 - \ell^2)},$$

which goes to 0 as $\ell \to \infty$.

Hence density of $P_\ell = 0$.

Finally, since $\{p : \text{charpoly}(T_p) \text{ is reducible over } \mathbf{Q}\} \subset \{p : \text{charpoly}(T_p) \text{ is reducible over } \mathbb{F}_\ell\}$. We have the density of $\{p : \text{charpoly}(T_p) \text{ is reducible over } \mathbf{Q}\} = 0$. This concludes the proof. $\square$

3 Remarks

Basically, I expect the same method generalizes to higher weight and higher level, by working with mod λ representations.

4 Acknowledgement

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5 References