HECKE: The Modular Forms Calculator

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What is HECKE?

`HECKE` is a program for computing with modular forms. It implements algorithms of Cremona, Hijikata, Merel, Mestre-Oesterlé, Shimura, and others. It is completely *free* software, currently available only for Linux machines at

`http://shimura.math.berkeley.edu/~was/Tables/hecke.html`

Unfortunately it is still in development, and quite a bit of work remains to be done[1]. I should emphasize that `HECKE` is *not* yet "rock solid" and ready for general release, so you should be especially skeptical of the results it gives and aware that the interface is at certain points very primitive. Nonetheless, `HECKE` is capable of many computations which aren't currently possible in any other single, integrated, publicly available, package. The main drawbacks are that the interface may be awkward and certain parts of the implementation have not been properly optimized.

What can HECKE do?

`HECKE` consists both of a `C++` library and an interactive calculator. Most of the following is implemented.

- **Modular forms and Hecke operators:** Computations on the spaces $M_k(\Gamma_1(N), \varepsilon)$, $k \geq 2$, over cyclotomic and finite fields. Functions include:
  - Computation of bases of newforms. Within computational limits, the level, weight, and character can be pretty much arbitrary, with the restriction that $k \geq 2$ be an integer. Furthermore, all eigenforms are computed, *not* just the ones with eigenvalues in $\mathbf{Q}$.
  - Exact computation of the rational numbers $L(M_f, i)/\Omega_i$ where M_f is a complex torus attached to f and Ω_i is a certain volume.
  - Optimal quotients A_f of $J_0(N)$ associated to newforms. (An optimal quotient is a quotient $J_0(N) \to A_f$ with connected kernel.)
    1. The modular degree and structure of the canonical polarization.
    2. Congruences.
    3. Order of image of $(0) - (\infty)$.
    4. Upper bound on torsion.
    5. Tamagawa numbers of semistable quotients of $J_0(N)$ (currently only for N prime).
  - Discriminants of Hecke algebras.

[1] Hecke is being written by me, with advice from K. Buzzard and H.W. Lenstra.

- Numerical computation of special values and period lattices of forms of even weight $k \geq 2$, in many (but not all) cases. When f has rational Fourier coefficients, computation of the invariants of the associated elliptic curve over $\mathbf{R}$.

- **Formulas:** The classical formulas, such as the numbers of cusps on modular curves, dimensions of spaces of cusps forms, and computation of $\dim S_k(\Gamma_1(N), \varepsilon)$ for $k \geq 2$ and ε a Dirichlet character modulo N (using the Hijikata trace formula).
- **Character groups of tori:** Action of Hecke operators on the character group associated to $J_0(p)$ (using the Mestre-Oesterlé graph method). The matrices attained in this way are very sparse.
- **Tables:** Functions for making tables of eigenforms.
- **More:** And much more...

Why does `HECKE` exist?

`HECKE` grew out of work on my thesis which involves computing special values of L-functions, congruences, and verifying modularity of certain Galois representations. In a sense, `HECKE` is also the program I wish had existed when I was taking my first modular forms course and wanted to see lots of concrete examples of modular forms. (Some of the tables computed using `HECKE` can be found at `http://shimura.math.berkeley.edu/~was/Tables`.)

Guided tour

In this guided tour, you will see how to use `HECKE` to compute the action of Hecke operators, bases of eigenforms, and obtain information about special values of L-functions.

Starting `HECKE`.

To start `HECKE`, type `hecke` at the command line. You will see something like

```
# hecke
  HECKE:  Modular Forms Calculator   Version 0.4 (June 14, 1999)
                    William A. Stein
Send bug reports and suggestions to was@math.berkeley.edu.
Type ? for help.
HECKE>
```

Typing `?` gives a list of "modes" which include:

```
  calc:      Motive calculator
  exsymbols: Extended modular symbols mode
  formulas:  Formula calculator
```

```
graphs:    Monodromy pairing calculator
msymbols:  Modular symbols calculator
tables:    Table making routines
```

Modular forms and Hecke operators calculator.

Type `msymbols` to start the modular forms and Hecke operators calculator. You will be asked for several bits of information which define the space on which to work. Answer as follows:

```
level N = 389
character chi = 0
weight k = 2
```

After a brief computation the calculator interface will print some information about $M_2(\Gamma_0(389))$ and await your command.

```
----------------------------------------------------------------
Current space:   M_2(Gamma_0(389); Q)^+, dim=33
Hecke action on: V=M_2, dim=33
----------------------------------------------------------------
M_2(389) ?
```

The help system is similar to that in PARI. Typing `?` gives a list of subtopics.

```
1: computing OPERATORS
2: setting current SPACE
3: cutting out SUBSPACES
4: computing BASIS
5: CONVERSIONS between representations
6: arithmetic INVARIANTS of torus A_V
7: INVARIANTS of Hecke algebra
8: OPTIONS
```

To get an idea of what $M_2(\Gamma_0(389))$ looks like, compute the characteristic polynomials of several Hecke operators T_n. Type `t` then enter a positive integer n.

```
? t
Tn: Enter values of n, then q when done.
2
f2=(x-3)*(x + 2)*(x^2 -2)*(x^3 -4*x -2)*
   (x^20 -3*x^19 -29*x^18 + 91*x^17 + 338*x^16 -1130*x^15
    -2023*x^14 + 7432*x^13 + 6558*x^12 -28021*x^11 -10909*x^10
    + 61267*x^9 + 6954*x^8 -74752*x^7 + 1407*x^6 + 46330*x^5
    -1087*x^4 -12558*x^3 -942*x^2 + 960*x + 148)*
   (x^6 + 3*x^5 -2*x^4 -8*x^3 + 2*x^2 + 4*x -1);
q
```

Let's compute the action of a few Hecke operators on the dimension two factor. Type `subeigenpoly`, then select the dimension two factor:

```
M_2(389) ? subeigenpoly
[...]
n = 2                             <---- you type this
Choose one of the following factors.
     1: x+2
     2: x-3
     3: x^2-2
     4: x^3-4*x-2
     5: x^20-3*x^19-29*x^18+91*x^17+338*x^16-1130*x^15-2023*x^14+
     7432*x^13+6558*x^12-28021*x^11-10909*x^10+61267*x^9+6954*x^8-
     74752*x^7+1407*x^6+46330*x^5-1087*x^4-12558*x^3-942*x^2+960*x+148
     6: x^6+3*x^5-2*x^4-8*x^3+2*x^2+4*x-1
     7: ALL factors
Select a factor: 3               <---- you type this
```

When the `M_2(389) ?` prompt appears, type `opmatrix` to turn on matrix display and `opcharpoly` to turn off computation of characteristic polynomials. Now you can compute matrices which represent the Hecke operators on this dimension two space:

```
M_2(389) ? t
2
T2=[2,1;-2,-2];
3
T3=[0,1;-2,-4];
6
T6=[-2,-2;4,6];
```

Let A denote the corresponding dimension two optimal quotient of $J_0(389)$. To compute the BSD value $L(A,1)/\Omega_A$, type `torusbsd`. `HECKE` outputs 0 along with the first few terms of the q-expansion of f and the discriminant of the ring $\mathbf{Z}[\ldots, a_n, \ldots]$. The sign in the functional equation for the L-function is minus the sign of the Atkin-Lehner involution W_{389}. To compute this involution, type `actatkin` and then enter `389` for p. `HECKE` compute that $W_{389} = +1$ on A, so the sign in the functional equation is -1 and $L(A,1)$ is forced to vanish.

To obtain the q-expansion of a normalized eigenform in our dimension two space, type `basisnew` then `n=7`. The result is

```
s1=t^2-2;  s=Mod(t,t^2-2);
f1 = q + (s)*q^2 + (s-2)*q^3 + -1*q^5 + (-2*s+2)*q^6 + (-2*s-1)*q^7 + O(q^8);
```

which means that a normalized newform is

$$f_1 = q + \sqrt{2}q^2 + (\sqrt{2}-2)q^3 - q^5 + (-2\sqrt{2}+2)q^6 + (-2\sqrt{2}-1)q^7 + \cdots$$

To compute the discriminant of the Hecke algebra $\mathbf{T}$, type `heckedisc`. `HECKE` computes the discriminant of the $\mathbf{Z}$-module generated by $T_1, \ldots, T$ and finds:

592456554486106225601956409404798293104261020095616213409857536000000

$= 2^{53} \cdot 3^4 \cdot 5^6 \cdot 31^2 \cdot 37 \cdot 97^2 \cdot 389 \cdot 3881 \cdot 215517113148241 \cdot 477439237737571441$

This is only known example in which $p \mid \mathrm{disc}(\mathbf{T}_{389})$ (there are no other such $p < 12000$).

Nontrivial character and weight.

Next, compute a basis of eigenforms for $S_4(\Gamma_0(13), \varepsilon)$ where $\varepsilon : (\mathbf{Z}/13\mathbf{Z}) \to \mathbf{C}^*$ is a character whose image has order 3. Type `x` to quit computing on $M_2(389)$, type `msymbols` again and enter `N = 13`, `chi = 3`, and `k = 4`. In a second, the status display will appear:

```
-----------------------------------------------------------------------
Current space:   M_4(Gamma_0(13), eps=[3]; Q[a]/(a^2-a+1))^+, dim=5
Hecke action on: V=M_4, dim=5
-----------------------------------------------------------------------
M_4(13) ?
```

(In the version you're using, the quadratic polynomial might be in x instead of a.) Type `basisnew`, then `n = 3` to get the first 3 terms of the q-expansions of a basis of newforms. Note: only one representative from each Galois conjugacy class of newforms is provided. The output is

```
f1 = q + (-4*a)*q^2 + (-2*a)*q^3 + O(q^4);
s2=t^2+(-5*a)*t+(2*a-2); s=Mod(t,t^2+(-5*a)*t+(2*a-2));
f2 = q + (s)*q^2 + ((-3)*s+(5*a))*q^3 + O(q^4);
```

This means that there are two (conjugacy classes of) eigenforms f_1 and f_2. The first is $f_1 = q - 4aq^2 - 2aq^3 + \cdots$ where a is a primitive cube root of 1, and the second is $f_2 = q+sq^2+(-3s+5a)q^3+\cdots$ where s is a root of $t^2-5at+2a-2 = 0$.

To work in fields of characteristic other than 0, use the extended mode by typing `exsymbols` instead of `msymbols` at the `HECKE>` prompt.

Motives associated to modular forms.

The `msymbols` mode is useful for computing basis of eigenforms and the action of Hecke operators on rather general spaces of modular forms. It is less useful for computing specific information about the structure of $J_0(N)$. For that, use the `calc` mode. Type `x` to get to the `HECKE>` prompt, then type `calc`. When asked if you want to work in the fast $+1$ quotient, typing `n`. (If you type `yes`, many computations will be orders of magnitude faster, but are likely to be wrong by a power of 2.) The basic syntax of a `calc` mode command is as follows:

[level]k[weight][isogeny class].[command][(arguments)]

Omitting the weight part of the command is the same as specifying $k = 2$. Type `125` to obtain a list of optimal quotients of $J_0(125)$.

```
? 125
****** SUMMARIZE LEVEL.
125k2:    dim    W
 A        2      +
 B        2      -
 C        4      -
```

This means that $J_0(125) \sim A \times B \times C$ where A, B, C are abelian varieties of dimensions 2, 2, and 4. We can compute $L(A, 1)/\Omega_A$, $L(B, 1)/\Omega_B$ and $L(C, 1)/\Omega_C$:

```
? 125A.bsdratio
0
? 125B.bsdratio
2^2/5
? 125C.bsdratio
1/5
```

The signs in the `W` column above give the signs of the Atkin-Lehner involution W_{125}. Thus one expects, because the level is low, that $J(\mathbf{Q}) \otimes \mathbf{Q} \cong A(\mathbf{Q}) \otimes \mathbf{Q} \approx \mathbf{Q} \oplus \mathbf{Q}$. This is in fact the case, though we will not prove it here. (I haven't yet implemented a function for computation $L'(A, 1)$, so the rank can't yet be bounded from within `HECKE`.) What about the torsion? Type `125.torsionbound(13)` to get an upper bound on the torsion subgroup of $J_0(125)$. Then type `125.cusporder` to compute the order of $(0) - (\infty) \in J_0(125)(\mathbf{Q})$.

```
? 125.torsionbound(13)
5^2
? 125.cusporder
5^2
```

We're lucky – the lower and upper bounds match up and we conclude that $J(\mathbf{Q}) \approx \mathbf{Z}^2 \oplus (\mathbf{Z}/25\mathbf{Z})$. Next type `125A.intersection(B)` to obtain the structure of the finite abelian group $A' \cap B' \subset J$, where A', B' are the abelian varieties dual to A and B. The answer `[2,2,2,2]` indicates that the intersection is $(\mathbf{Z}/2\mathbf{Z})^4$. This implies that the corresponding newforms satisfy a congruence in characteristic 2. To exit `calc` mode, type `\q`.

This tutorial has barely scratched the surface of what is possible using `HECKE`. If you are interesting in learning more, talk to me.