Constructing the Surreal Numbers

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History of Surreal Numbers

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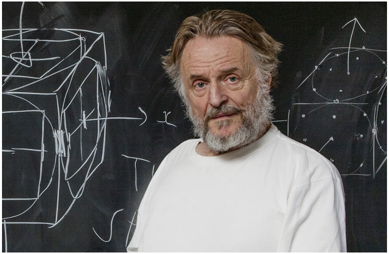

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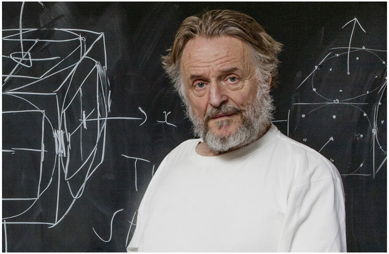
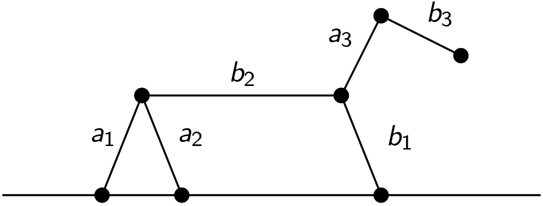
- “All Numbers Great and Small” by John Conway in 1972
- *Surreal Numbers* by Donald Knuth in 1974




Review of *Aerion*

Recall the following hackenbush game between players Left (who can remove pieces a_1, a_2, a_3) and Right (who can remove pieces b_1, b_2, b_3):



From Hackenbush to Ordered Pairs

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Let L_1, L_2, and L_3 denote the results of player Left exercising options a_1, a_2, and a_3, respectively, and let R_1, R_2, and R_3 denote the results of player Right exercising options b_1, b_2, and b_3, respectively.

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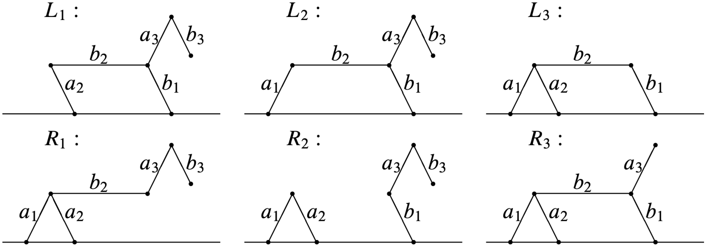

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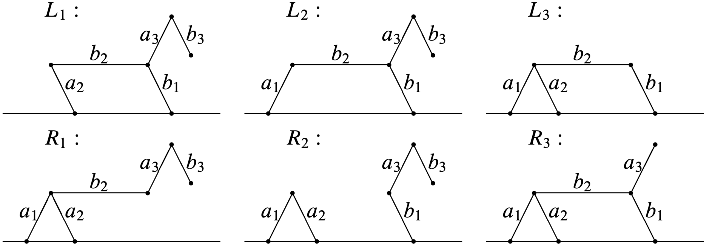

We can represent the *Aerion* game with the ordered tuple of sets

$$Aerion = (\{L_1, L_2, L_3\}, \{R_1, R_2, R_3\}).$$

Sequences of Games

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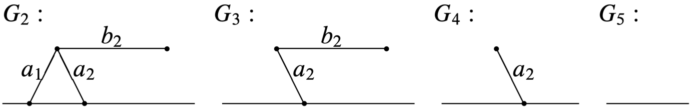
So, a particular "game-play" of Aerion is the sequence

$$G = (G_0, G_1, G_2, G_3, G_4, G_5),$$

where $G_0 = Aerion$, $G_1 = L3$, and



Inductive Collection

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We say that a collection of ordered pairs of sets is *inductive* if it has the property that whenever $\mathcal{L}$ and $\mathcal{R}$ are subsets of the collection, the ordered pair $(\mathcal{L}, \mathcal{R})$ is an element of the collection.

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Example

Let Ω be an inductive collection and suppose $\{1, 2, 3\} \subset \Omega$. Then

$$(\{1, 2\}, \{3\}) \in \Omega.$$

Categorical Definition of Ω

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Axioms

- **Axiom 24.1** The collection Ω is inductive.
- **Axiom 24.2** The only inductive subcollection of Ω is Ω itself.

Building Games

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We will build elements of Ω in "generations," writing Ω_n for the collection of games in the nth generation. Elements of Ω_n can be built using sets whose elements are in previous generations.

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Then we can define

$$\begin{aligned} G(1) &= (\{G(0)\}, \emptyset) \\ G(-1) &= (\emptyset, \{G(0)\}), \text{ and} \\ N(1) &= (\{G(0)\}, \{G(0)\}). \end{aligned}$$

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Thus we have that $\Omega_1 = \{G(1), G(-1), N(1)\}$.

We can continue, but it turns out that $|\Omega_2| = 252$, and clearly, generations continue to grow in size.

An Order Relation on Ω

Definition 24.3.

Let $G = (\mathcal{L}_G, \mathcal{R}_G) \in \Omega$ and $H = (\mathcal{L}_H, \mathcal{R}_H) \in \Omega$. We say that G is "less than or similar" to H and write $G \lesssim H$ if, and only if:

- There is no $L \in \mathcal{L}_G$ for which $H \lesssim L$.
- There is no $R \in \mathcal{R}_H$ for which $R \lesssim G$.

If G is not less than or similar to H, we write $G \not\lesssim H$.

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Proposition 24.4.

For any $\mathcal{X}, \mathcal{Y} \in \Omega$, we have that $(\emptyset, \mathcal{X}) \lesssim (\mathcal{Y}, \emptyset)$.

Addition in Ω

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Now we define an addition operation on Ω.

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Definition 24.22.

Let $G = (\mathcal{L}_G, \mathcal{R}_G) \in \Omega$ and $H = (\mathcal{L}_H, \mathcal{R}_H) \in \Omega$. We define the sum of G and H recursively as

$$G + H = ((G + \mathcal{L}_H) \cup (\mathcal{L}_G + H), (G + \mathcal{R}_H) \cup (\mathcal{R}_G + H)).$$

Here, as customary, the sum of a game and a collection of games is simply the collection of the appropriate sums; for example,

$$G + \mathcal{L}_H = \{G + L_H \mid L_H \in \mathcal{L}_H\}.$$

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Theorem 24.23.

Addition on Ω is a closed operation, is commutative, is associative, has the identity property, and has the additive inverse property (up to similarity).

Multiplication in Ω

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Definition 24.26.

Let $G = (\mathcal{L}_G, \mathcal{R}_G) \in \Omega$ and $H = (\mathcal{L}_H, \mathcal{R}_H) \in \Omega$. We define the product of G and H recursively as

$$\begin{aligned} G \cdot H = &((G \cdot \mathcal{L}_H + \mathcal{L}_G \cdot H - \mathcal{L}_G \cdot \mathcal{L}_H) \cup (G \cdot \mathcal{R}_H + \mathcal{R}_G \cdot H - \mathcal{R}_G \cdot \mathcal{R}_H), \\ &(G \cdot \mathcal{R}_H + \mathcal{L}_G \cdot H - \mathcal{L}_G \cdot \mathcal{R}_H) \cup (G \cdot \mathcal{L}_H + \mathcal{R}_G \cdot H - \mathcal{R}_G \cdot \mathcal{L}_H)). \end{aligned}$$

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Definition 24.7.

The collection of numeric games Γ consists of games $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L} \subseteq \Gamma$, $\mathcal{R} \subseteq \Gamma$, and there is no $L \in \mathcal{L}$ and $R \in \mathcal{R}$ for which $R \lesssim L$.

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Proposition 24.8.

For any $\mathcal{X} \in \Gamma$, we have $(\emptyset, \mathcal{X}) \in \Gamma$ and $(\mathcal{X}, \emptyset) \in \Gamma$.

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Write $\Gamma_n = \Gamma \cap \Omega_n$.

More Numeric Games

Using Definition 24.7 we can get

$$\Gamma_0 = \{G(0)\}, \text{and } \Gamma_1 = \{G(-1), G(1)\}.$$

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It turns out that $|\Gamma_2| = 17$. Here are four elements of Γ_2:

$$G(-2) = (\emptyset, \{G(-1), G(0)\});$$
$$G\left(-\frac{1}{2}\right) = (\{G(-1)\}, \{G(0)\});$$
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We will see shortly why we only have to consider these four!

Extending the Relation on Ω

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Definition 24.5.

Suppose that $G \in \Omega$ and $H \in \Omega$. We say that:

- G is "similar" to H and write $G \approx H$, if $G \lesssim H$ and $H \lesssim G$.
- G is "greater than" or similar to H and write $G \gtrsim H$ if $H \lesssim G$.
- G is "less than" H and write $G < H$, if $G \lesssim H$ and $G \not\approx H$.
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Proposition 24.12 and 24.14.

Let G, H, and K be arbitrary games, and suppose that $G \lesssim H$ and $H \lesssim K$. Then we also have $G \lesssim G$ and $G \lesssim K$.

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Corollary 24.20.

The relation $\approx$ of similarity is an equivalence relation on the collection of games Ω.

Defining Surreal Numbers

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The similarity relation partitions Ω into equivalence classes, but we are only interested in the equivalence classes within Γ.

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We denote the equivalence class of a game G by $[G]$; that is,

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Definition 24.21.

The equivalence classes of the similarity relation in Γ are called *surreal numbers*. The collection of surreal numbers is denoted by $\mathbb{S}$.

An Example of Equivalence Relation (Proposition 24.9)

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G	$G \in \dots$	$\dots \in [G]$
$G(0)$	Γ_0	$(\{G(-1)\}, \emptyset)$, $(\emptyset, \{G(1)\})$, $(\{G(-1)\}, \{G(1)\})$
$G(1)$	Γ_1	$(\{G(-1), G(0)\}, \emptyset)$
$G(-1)$	Γ_1	$(\emptyset, \{G(0), G(1)\})$
$G(2)$	Γ_2	$(\{G(1)\}, \emptyset, \emptyset)$ $(\{G(-1), G(1)\}, \emptyset)$, $(\{G(-1), G(0), G(1)\}, \emptyset)$
$G(-2)$	Γ_2	$(\emptyset, \{G(-1)\})$ $(\emptyset, \{G(-1), G(1)\})$, $(\emptyset, \{G(-1), G(0), G(1)\})$
$G(1/2)$	Γ_2	$(\{G(-1), G(0)\}, \{G(1)\})$
$G(-1/2)$	Γ_2	$(\{G(-1)\}, \{G(0), G(1)\})$

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Definition 24.27.

Let $x_\mathbb{S}$ and $y_\mathbb{S}$ be surreal numbers, and suppose that $G \in x_\mathbb{S}$ and $H \in y_\mathbb{S}$. We define surreal addition and multiplication as

- $x_\mathbb{S} + y_\mathbb{S} = [G + H]$; and
- $x_\mathbb{S} \cdot y_\mathbb{S} = [G \cdot H]$.

Furthermore, we say that $x_\mathbb{S}$ is positive if $G > G(0)$ and that it it is negative if $G < G(0)$; we let $P_\mathbb{S}$ denote the collection of positive surreal numbers.

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Theorem 24.28.

The surreal number system $(\mathbb{S}, +, \cdot, P_{\mathbb{S}})$ is an ordered field.

Interesting Results

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Theorem 24.29.

The surreal number system $(\mathbb{S}, +, \cdot, P_{\mathbb{S}})$ is the largest ordered field; that is, for every ordered field system $(F, +_F, \cdot_F, P_F)$, there exist a subfield $S \subseteq \mathbb{S}$ for which the systems $(F, +_F, \cdot_F, P_F)$ and $(S, +_S, \cdot, P_S)$ are isomorphic.

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As a Corollary to Axiom 24.2 we have,

Theorem 24.10. (Conway's Induction Principle)

Suppose that $P(G)$ is a predicate that becomes a statement for all $G \in \Omega$. If the implication

$$\left(\bigwedge_{K \in \mathcal{L} \cup \mathcal{R}} P(K) \right) \implies P(H)$$

holds for all $H = (\mathcal{L}, \mathcal{R}) \in \Omega$, then $P(G)$ is true for every $G \in \Omega$.