$\mathbf{R}$-analysis (Brown)

$f \in C^0\{a\}$ short for f continuous at a
intervals $(x-r, x+r)$ sometimes notated $(x \pm r)$ or B_x^r
subsets proper unless $\subseteq$

1.1.

1.1.1 Show equivalance of the following:

- (a) $N \in Op(x)$ (a neighborhood of x)
- (b) $[x \pm \delta] \subset N$
- (c) $[x \pm n^{-1}] \subset N$

Proof. $\exists \delta' : (x-\delta', x+\delta') \subset N$.
Defining $\delta = \frac{1}{3}\delta', [x-\delta, x+\delta] \subset x-\delta', x+\delta') \subset N$
$(a) \implies (b)$

Fixing $\delta < 1, \exists n : 1 < n\delta \implies n^{-1} < \delta$ and $[x \pm n^{-1}] \subset [x \pm \delta] \subset N$
$(b) \implies (c)$
$(c) \implies (a)$
□

1.1.2 Prove $Int(\mathbf{FinSet}) = \varnothing$.

Proof. let x, y be distinct points and fix $\epsilon = \frac{1}{3}\inf_{x,y\in S} d(x,y) > 0$.
$d(x, x+\epsilon) < \inf_S d(x,y) \implies x \pm \epsilon \notin S$. □

1.1.3 Prove $Int(A \cap B) = IntA \cap IntB$.

Proof. Holds trivially for empty sets. elements satifying $x+\epsilon_2 \leq x+\epsilon' \in IntA, x+\epsilon'' \in IntB$ are exactly $IntA \cap IntB$. The interior of the intersection are elements $x+\epsilon_1 : \epsilon_1 = \inf\{\epsilon', \epsilon''\}$ ie simultaneously interior $A \cap B$
Clearly $x+\epsilon_2 \leq x+\epsilon_1 \implies IntA \cap IntB \subset Int(A \cap B)$ and for nonempty $A, B, Int(A \cap B) \subset IntA \cap IntB$. Therefore, $Int(A \cap B) = IntA \cap IntB$ □

1.1.4 Does the converse, $Int(A \cup B) = IntA \cup IntB$ hold? No.
Consider $A = [0,1), B = [1,2] : Int(A \cup B) = (0,2) \neq IntA \cup IntB = (0,2) - \{1\}$

1.1.5 Does $Int \bigcap A_i = \bigcap IntA_i$?

Proof. Use induction. from 1.1.3, $Int(A \cap B) = IntA \cap IntB$.
$Int\left(A_i \cap (\bigcap A_{i-1})\right) = Int(\bigcap A_i)$. Conversely, $IntA_i \cap Int \bigcap A_{i-1} = \bigcap IntA_i$. □

1.1.6-8. Show neighborhoods preserve addition, multiplication and inverse operations.

Proof. 1.1.6 (addition) Setting

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{3}$$

$(a+\epsilon_1) + (b+\epsilon_2) = (a+b) + \frac{2}{3}\epsilon < (a+b) + \epsilon$ □
1.1.7 (multiplication) [works for $\epsilon \leq 1$] Setting

$$\epsilon_1 = \epsilon_2 = \inf\left\{????, \frac{\epsilon}{2}\inf\left\{1, \frac{1}{a+b}\right\}\right\}$$

$$(a+\epsilon_1)(b+\epsilon_2) = \begin{cases} ab + \frac{1}{2}\epsilon + \frac{1}{4}\epsilon^2 < ab + \frac{3}{4}\epsilon & \frac{1}{a+b} \le 1 \\ ab + \frac{\epsilon}{2}(a+b) + \frac{1}{4}\epsilon^2 < ab + \frac{\epsilon}{2} + \frac{1}{4}\epsilon^2 & \frac{1}{a+b} > 1 \end{cases} \quad \square$$

1.1.8 (multiplicative inverses) □

1.1.9 Prove $Int(IntA) = IntA$.

Proof. asdf □

1.1.10 Show there are exactly 14 subsets generated by the complementation and Int operators from base sets A_1, A_2, A_3.

1.2. f *continuous* at a: $\forall V_{f(a)} \exists U_a : f[U_a] \subset V \iff \lim_{x\to a} f(x) = f(a)$

1.2.1 For $f, g \in C^0\{a\}, \exists ? h, h'|a \in C^0 : h = f + g, h' = f * g \in C^0, \frac{f}{g}$.

Proof. Fix $\delta := \min\{\delta', \delta''\}, h(x) := f(x) + g(x)$.
Then $h(x+\delta) = f(x+\delta) + g(x+\delta) \le f(x) + g(x) + \epsilon' + \epsilon''$ □

1.2.2 Prove squeeze, demonstrate with $x \sin \frac{1}{x}$.

Proof. asdf □

1.2.3 Prove $f|_{A\cap N} \in C^0\{a\} \implies f \in C^0\{a\}$.

Proof. asdf □

1.2.4 Show $f(x) := x, x \in [0,1); f(x) := x - 1, x \in [2,3]$ *continuous and injective but* $f^{-1} \notin C^0\{1\}$.

Proof. asdf □

1.2.5 Prove monotone bij function $f : [a,b] \to [c,d]$ *continuous.*

Proof. asdf □

1.2.6 $f \in End(\mathbf{R})$, *show* $f \in C^0\mathbf{R} \iff f^{-1}IntA \subset Intf^{-1}[A]$.

Proof. asdf □

1.2.7 Show equivalance of the following:

- (a) $f \in C^0\{0\}$
- (b) $\forall \epsilon > 0, \exists \delta : f(a-\delta, a+\delta) \subset (f(a) - \epsilon, f(a) + \epsilon)$
- (c) $\forall n \in \mathbf{N}, \exists m : f(a - m^{-1}, a + m^{-1}) \subset (f(a) - n^{-1}, f(a) + n^{-1})$

Proof. asdf □

1.2.8 (Gluing theorem) Suppose there exists a function.

$$f : A \to \mathbf{R} : f|_{A_1,A_2} \in C^0\{0\}, A = \bigcup A_i, a \in \bigcap A_i$$

Proof. poof □

1.2.9 Prove $\left|\hom_{C^0}[\mathbf{I}, \mathbf{R}]\right| \notin \aleph_0$.

Proof. Try iso from R to I to R then compound map is an R-automorphism, the constant functions should be uncountable and continuous. □

1.3 More Int, Ext and Fr.