Using Physical Simulation to Plot Graphs

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Overview

1 Motivation
- Background Information
- Leading Questions
- The Problem

2 The Physical Model
- Steps Toward the Goal
- The Goal

3 Future Work
- Computation
- Investigation

4 Questions

Graph Theory

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Definition 1

Given a set $\mathcal{W}$, a **graph** G is a proper subset of the Cartesian product, $\mathcal{W} \times \mathcal{W}$, such that for any $(A, B) \in G$, it is true that $A \neq B$ and $(B, A) \in G$.

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Example 2

Let $\mathcal{W} = \{0, 1, 2, 3\}$. Then

$$G = \{(0,1), (1,0), (0,2), (2,0), (1,2), (2,1), (1,3), (3,1)\}$$

is a graph.

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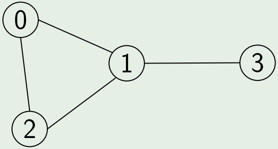

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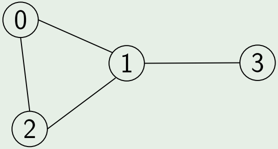

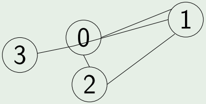

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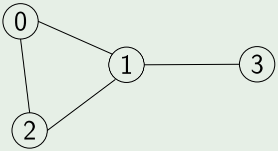
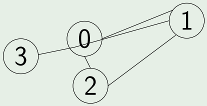
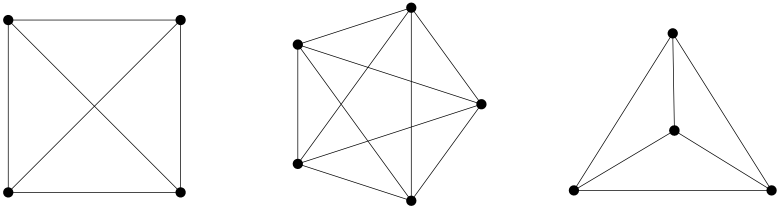




Definition 3

A graph is **planar** if it can be plotted in the plane with no intersecting edges.

Are these graphs different?



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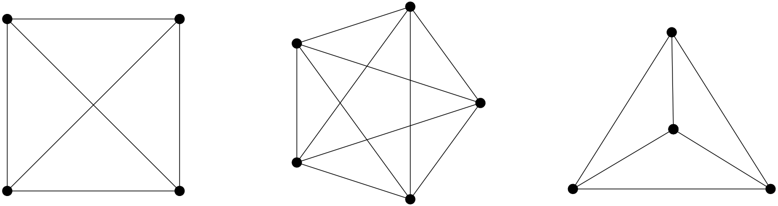

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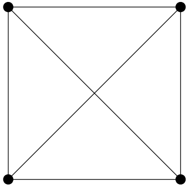

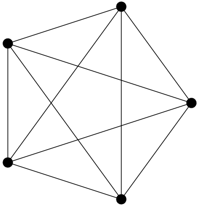

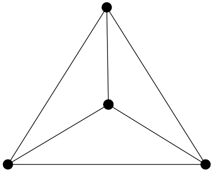

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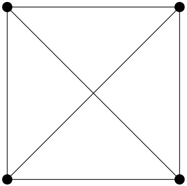

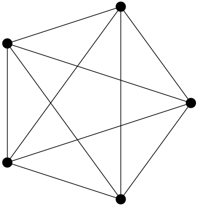

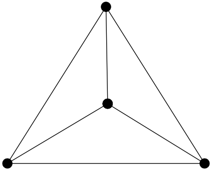

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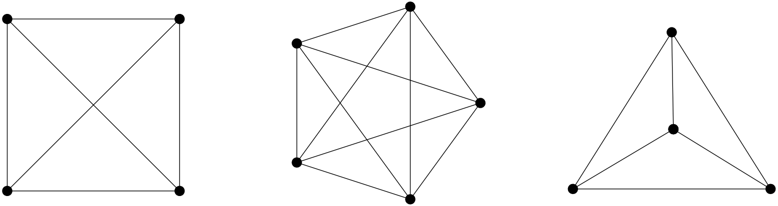

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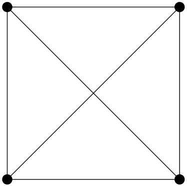

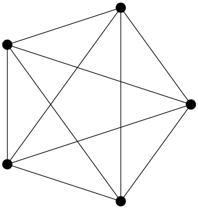

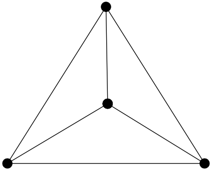

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- K_5 and $K_{3,3}$ as subgraphs

How can we efficiently plot graphs in a visually satisfactory way?

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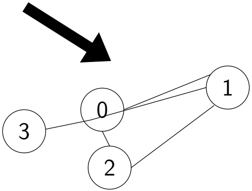

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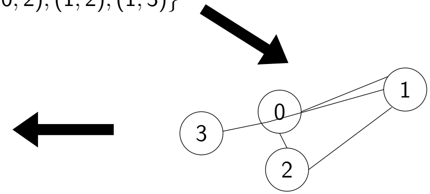

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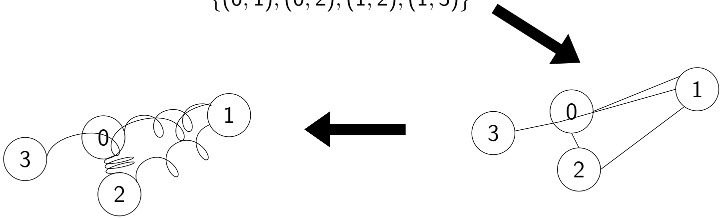

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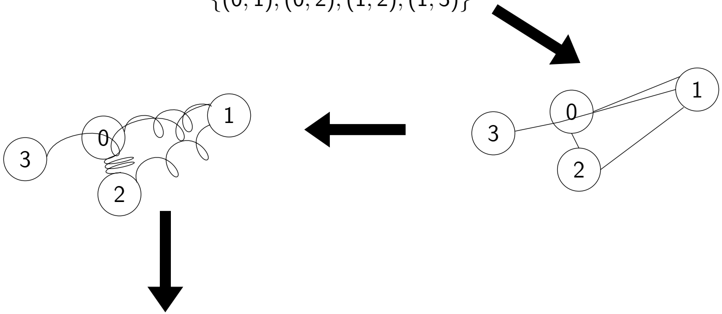

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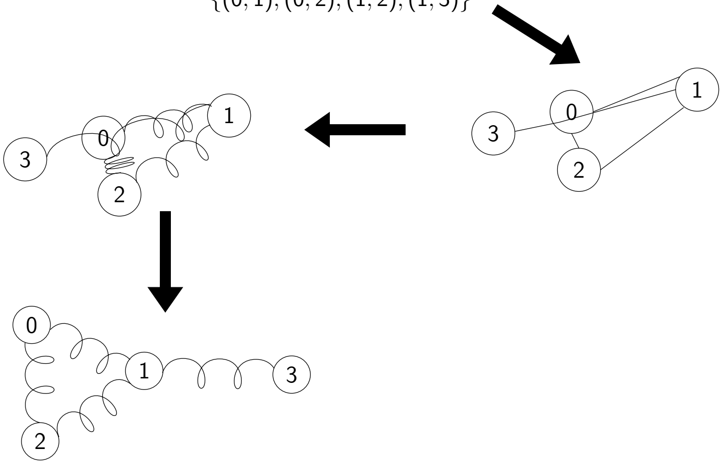

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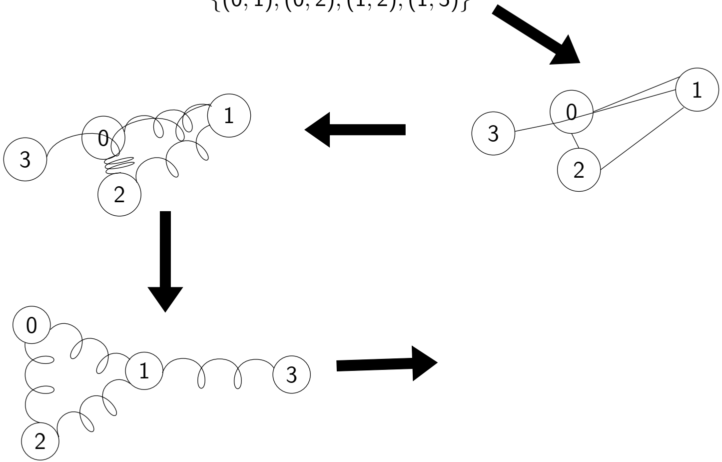

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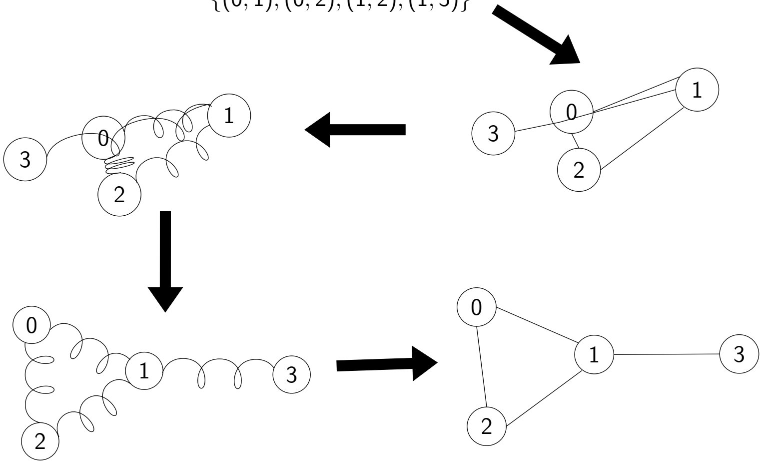

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Definition 4

The **degree** of node A in a graph G is the number of nodes that are connected to A, and is notated $\deg(A)$.

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Let Δt be the time step of the simulation. For each node A, we can use the net force on A to recursively approximate its $j + 1^{\text{st}}$ displacement, $\Delta\vec{d}_{j+1}$:

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which implies that

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- `Graph` and `DiGraph` wrapper classes

Physical Investigation

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- Animation

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- Long term behavior of complex systems

Questions?